Abstract

We give a generalized version of uncertainty principle, and apply it to the study of regularization properties of solutions to kinetic equations. In particular, both linearized and nonlinear space inhomogeneous Boltzmann equations without Grad’s cutoff assumption are considered. To cite this article: R. Alexandre et al., C. R. Acad. Sci. Paris, Ser. I 345 (2007).

Résumé


Version française abrégée

En vue d’étudier les propriétés d’hypoellipticité des équations cinétiques de type Boltzmann, dans le cadre des noyaux singuliers, c’est-à-dire sans faire l’hypothèse de troncature angulaire de Grad, nous prouvons tout d’abord une version adaptée du principe d’incertitude de Fefferman.

Le cadre de ce principe est le suivant. On considère deux fonctions positives $a^+$ et $a^-$ de la variable $v \in \mathbb{R}^n$, avec $a^- \in L^\infty(\mathbb{R}^n)$. Pour $0 < s < 1$, on associe à l’opérateur $a_s(D_v) = |D_v|^{2s}$, la fonction $	ilde{a} = \tilde{a}(r) = r^{-2s}$ de la variable réelle $r > 0$. Pour un cube $Q \subset \mathbb{R}^n$ donné, de longueur de côté $l(Q)$, on désigne par $Q^*$ tout cube tel que $Q \subset Q^*$ et $l(Q^*) = 2l(Q)$. Pour un tel couple de cubes $(Q, Q^*)$, on leur associe le sous-ensemble suivant de $\mathbb{R}^n$

$$E(Q, Q^*) \equiv \{ v \in Q^*, a^+(v) \geq ||a^-||_{L^\infty(Q)} - \tilde{a}(l(Q)) \}.$$
Notre hypothèse principale est alors la suivante : il existe une constante \( \kappa > 0 \) telle que
\[
\inf_{(Q, Q^*), l(Q^*)=2l(Q)} \frac{|E(Q, Q^*)|}{|Q^*|} \geq \kappa.
\] (H)

On a alors le théorème suivant :

**Théorème 0.1.** Avec les notations et hypothèses précédentes, il existe une constante \( C = C(\kappa, n) \) telle que
\[
\int_{\mathbb{R}^n} a^-(v) |f(v)|^2 \, dv \leq C \int_{\mathbb{R}^n} \{ |a_{1/2}^{1/2} (D_v) f(v)|^2 + a^+(v) |f(v)|^2 \} \, dv,
\]

pour toute fonction \( f \in \mathcal{S}(\mathbb{R}^n) \).

A quelques modifications près, la même inégalité reste valable si l’opérateur \( a_{1/2}^{1/2} \) est remplacé par l’opérateur \( a_{1/\log} \equiv (\log(D_v))^s \). Comme application de ce principe, nous considérons le cadre des équations cinétiques, et nous limitons ici à un seul exemple, renvoyant à la version anglaise pour d’autres cas, et en particulier les équations de Boltzmann. On a alors le théorème suivant :

**Théorème 0.2.** Soit \( g \in H^{-s'}(\mathbb{R}^{2n+1}) \) pour un certain \( 0 \leq s' < 1 \), et soit \( f \in L^2(\mathbb{R}^{2n+1}) \) satisfaisant
\[
\partial_t f + v \cdot \nabla_x f = g \quad \text{in} \quad D'(\mathbb{R}^{2n+1}),
\]
on où \( (t, x, v) \in \mathbb{R}^{1+n+n} \). On suppose de plus que \( a_{1/2}^{1/2}(D_v) f \in L^2(\mathbb{R}^{2n+1}), 0 < s \leq 1 \). Alors
\[
\frac{1}{(1 + |v|^2)^{s'/2(s+1)}} \langle D_v \rangle f \quad \text{et} \quad \frac{1}{(1 + |v|^2)^{s'/2(s+1)}} \langle D_{1/2} \rangle f \in L^2(\mathbb{R}^{2n+1}).
\]

Pour plus de détails et d’autres résultats, nous renvoyons à la version anglaise. Tous les résultats présentés ici, ainsi que des extensions, seront détaillés dans un article à paraître [2].

1. Introduction and main results

Having in mind the study of regularity of weak solutions to Boltzmann type kinetic equations, by using tools from pseudo-differential operators theory, and more generally harmonic analysis, we shall first show a generalized version of Fefferman’s uncertainty principle. For further details on this principle, we refer to [4–8]. Let us note that another method is still available, in some cases, see [10].

For variable \( v \in \mathbb{R}^n \), let \( a^+(v) \) and \( a^-(v) \) be two nonnegative functions, with \( a^- \in L^\infty(\mathbb{R}^n) \). Let \( a_s(D_v) \equiv |D_v|^{2s} \) for \( 0 < s < 1 \). We associate, for variable \( r > 0 \), the function \( \tilde{a}(r) = r^{-2s} \).

For any cube \( Q \subset \mathbb{R}^n \), we denote by \( l(Q) \) its side length, and by \( Q^* \) any cube such that \( Q \subset Q^* \) with \( l(Q^*) = 2l(Q) \). A given couple of cubes \( (Q, Q^*) \) is then associated with the subset
\[
E(Q, Q^*) = \{ v \in Q^*, a^+(v) \geq \|a^-\|_{L^\infty(Q)} - \tilde{a}(l(Q)) \}.
\]

Our main assumption on the functions \( a^+ \) and \( a^- \) is then the following: we assume there exists a constant \( \kappa > 0 \) such that
\[
\inf_{(Q, Q^*), l(Q^*)=2l(Q)} \frac{|E(Q, Q^*)|}{|Q^*|} \geq \kappa.
\] (H)

**Theorem 1.1** (Uncertainty principle). Under the above notations and assumptions, in particular (H), there exists a constant \( C = C(\kappa, n) \), such that for all \( f \in \mathcal{S}(\mathbb{R}^n) \), one has
\[
\int_{\mathbb{R}^n} a^-(v) |f(v)|^2 \, dv \leq C \int_{\mathbb{R}^n} \{ |a_{1/2}^{1/2} (D_v) f(v)|^2 + a^+(v) |f(v)|^2 \} \, dv.
\]

A similar estimate holds also true, if the operator \( a_{1/2}^{1/2} \) is replaced by \( a_{s,\log}^{1/2} \equiv (\log(D_v))^s \) with \( s > \frac{1}{2} \).
Our main interest lies in the application of this generalized uncertainty principle to the study of hypoellipticity of kinetic equations. To compare our approach with [3], a typical example is given by the following result:

**Theorem 1.2.** Assume that \( g \in H^{-s'}(\mathbb{R}^{2n+1}) \), for some \( 0 \leq s' < 1 \) and \( f \in L^2(\mathbb{R}^{2n+1}) \) is a weak solution of the following kinetic equation

\[
\partial_t f + v \cdot \nabla_x f = g,
\]

where \((t, x, v) \in \mathbb{R}^{1+n+n}\). If furthermore \( a_s^{1/2}(D_x)f \in L^2(\mathbb{R}^{2n+1}), 0 < s \leq 1 \), then it follows that

\[
\frac{\langle D_x \rangle^{s(1-s')/(s+1)} f}{(1 + |v|^2)^{s'/2(s+1)}} \in L^2(\mathbb{R}^{2n+1}).
\]

As a first step to a regularity result for the full space inhomogeneous nonlinear Boltzmann equation, we now consider applications of the above hypoellipticity estimation and method to the linearized Boltzmann equation, in the case of non cutoff cross-sections. For further details on Boltzmann equation in this case, we refer to [11].

We shall work in dimension \( n = 3 \), though other dimensions could be considered as well. The usual Boltzmann quadratic operator \( Q(g, f) \) is given by:

\[
Q(g, f) = \int B(v - v_s, \sigma) \{g(v')f(v') - g(v_s)f(v_s)\} \, d\sigma \, dv_s.
\]

Above, \( \sigma \)-representation has been used for describing the post and pre collisional velocities. Furthermore, we choose the collision kernel \( B \) to be of Maxwellian molecule type, and corresponding to inverse power laws type potentials with small singularity. That is, we assume, for some constant \( K > 0 \) and some \( 0 < s < \frac{1}{2} \),

\[
B = b(\cos \theta) \sim K|\theta|^{-2-2s}, \quad \text{when} \ \theta \to 0,
\]

so that Grad’s cutoff assumption is not satisfied. Note that for the Maxwellian molecules, \( s = 1/4 \). The singularity of collision kernel (1) implies the following subelliptic estimate: If \( g \in L^1_2 \cap L \log L(\mathbb{R}^3_+ \) and \( g \geq 0 \), then there exists a constant \( C_g > 0 \) such that

\[
C_g^{-1} \left\| a_s^{1/2}(D_x)f \right\|_{L^2(\mathbb{R}^3)}^2 \leq \left( -Q(g, f), f \right)_{L^2(\mathbb{R}^3)} + C_g \|f\|_{L^2(\mathbb{R}^3)}^2,
\]

for any smooth function \( f \in H^2(\mathbb{R}^3) \). This subelliptic estimate was crucially used in the proof of the regularizing effects of the cross-sections on the solutions for the spatially homogeneous Boltzmann equations, cf. [1,9] and the references therein. In this work, we apply it to the spatially inhomogeneous problem through uncertainty principle.

The linearized Boltzmann operator around the normalized Maxwellian distribution \( \mu(v) = (2\pi)^{-3/2} e^{-|v|^2/2} \) is then given by \( Lf = Q(\mu, f) + Q(f, \mu) \), along with the following Cauchy problem

\[
\partial_t f + v \cdot \nabla_x f = Lf, \quad x, v \in \mathbb{R}^3, \ t > 0; \quad f|_{t=0} = f_0.
\]

**Theorem 1.3.** Assume that \( \langle v \rangle f_0 \in L^2(\mathbb{R}^3 \times \mathbb{R}^3) \) for all \( l \in \mathbb{N} \). Then there exists a unique weak solution \( f \) of Cauchy problem (2) such that

\[
\langle v \rangle f \in L^\infty \cap L^2([0, T[ \times \mathbb{R}^6), \quad \forall l \in \mathbb{N}, \ \forall T > 0.
\]

Furthermore, for all \( l \in \mathbb{N} \), for all \( 0 < \delta < T' < T \), it follows that

\[
\langle v \rangle f \in H^{+\infty}([\delta, T'[ \times \mathbb{R}^6).
\]

As regard to the full nonlinear space inhomogeneous Boltzmann equation

\[
\partial_t f + v \cdot \nabla_x f = Q(f, f), \quad x, v \in \mathbb{R}^3, \ t > 0.
\]

One has the following:
Theorem 1.4. Assume that $f \in L^\infty([0, T[a]; L^1_2 \cap L^\infty(\mathbb{R}^3 \times \mathbb{R}^3))$ is a weak solution of Eq. (3), and

$$(1 + |D_v|^2)^{s/2} f \in L^2_{loc}[0, T],$$

$$(1 + |D_v|^2)^{-s} Q(f, f) \in L^2([0, T[\times\mathbb{R}^6]).$$

Then, it follows that

$$(1 + |D_t|^2 + |D_x|^2)^{\frac{s(1-2s)}{2(1+s)}} f \in L^2_{loc}[0, T[\times\mathbb{R}^6).$$

Note carefully that in the above result, we have assumed the existence of weak solutions in the class of functions mentioned in the assumptions. Such an existence theorem is still not available at present, and it is currently under study. Further details of the above results will be given in [2].

2. Ideas of the proofs

2.1. Proof of Theorem 1.1

For $k \in \mathbb{Z}$, let $D_k$ be the collection of cubes with side length $2^{-k}$ and vertices located at points in $2^{-k}\mathbb{Z}^n$, and $\mathcal{D} = \bigcup_{k \in \mathbb{Z}} D_k$ the set of all dyadic cubes. If a cube $Q \in \mathcal{D}$ is obtained by division of a bigger dyadic cube $Q'$ with double side length, we call $Q'$ the mother cube of $Q$.

For a constant $A > 0$ and a function $a^- \in L^\infty(\mathbb{R}^n)$, a cube $Q$ is said to satisfy property $\mathcal{F}(A, a^-)$ if

$$\|a^-\|_{L^\infty(Q)} \leq A a^\ast(l(Q)).$$

Then, we denote by $\mathcal{M}(A, a^-)$ the set of all dyadic cubes $Q \in \mathcal{D}$ satisfying $\mathcal{F}(A, a^-)$ but not their mother cube. Clearly, if a cube $Q'$ satisfies $\mathcal{F}(A, a^-)$ then so does any cube $Q \subset Q'$. It follows that we have a nonoverlapping covering on $\mathbb{R}^n$ by the elements in $\mathcal{M}(A, a^-)$. For such a covering, another locally finite covering on $\mathbb{R}^n$ can be constructed, taking into account the hypothesis (H).

Proposition 2.1. Assume the hypothesis (H). Then there exists a constant $A_s$ depending only on $s$, such that for all $A \geq A_s$, for the collection of cubes $Q_j \in \mathcal{M}(A, a^-)$, there exists another collection of cubes $\tilde{Q}_j$ satisfying the following:

(i) $Q_j \subset \tilde{Q}_j \subset Q_j'$, where $Q_j'$ denotes a mother cube of $Q_j$;

(ii) There exists a dual pair of cubes $(Q, Q^s)$, $Q$ being of side length $\frac{1}{2} l(Q_j)$ such that

$$|\tilde{Q}_j \cap E_j(Q, Q^s)| \geq \frac{\kappa}{2} |Q_j|,$$

and

$$a^+(v) \geq C_1 a^\ast(l(Q_j)) \geq C_2 \|a^-\|_{L^\infty(Q_j)}, \quad v \in E_j(Q, Q^s),$$

where $C_1, C_2$ are the constants independents of $j$.

(iii) The collection $\{\tilde{Q}_j\}$ is a locally finite covering of $\mathbb{R}^n$, and the number of overlapping is order $O(\kappa^{-n})$.

The proof of this result follows by a detailed checking case by case, and is detailed in [2]. Once this result available the final proof of Theorem 1.1 follows by using the equivalent integral type definition of Sobolev spaces.

2.2. Proof of Theorem 1.2

The following estimate is seeked for:

$$\|A(v, D_t, D_x) f\|_{L^2(\mathbb{R}^{2n+1})} \leq C \{ \|D_x^{1/2}(D_v) f\|_{L^2(\mathbb{R}^{2n+1})}^2 + \|X_0 f\|_{H^{-s'}(\mathbb{R}^{2n+1})}^2 + \|f\|_{H^s(\mathbb{R}^{2n+1})}^2 \},$$

for all $f \in \mathcal{S}(\mathbb{R}^{2n+1})$, where $X_0 \equiv (\partial_t + v \cdot \nabla_x)$ and

$$A(v, D_t, D_x) f = \frac{(D_x)^{(1-s')/(s+1)} f}{(1 + |v|^2)^{s'/2(s+1)}} + \frac{(D_t)^{(1-s')/(s+1)} f}{(1 + |v|^2)^{s'/2(s+1)}}.$$
We denote by $\tau$, $\eta$ and $\xi$ the dual (Fourier) variables corresponding to $t$, $x$ and $v$ respectively. Choose $\chi(\tau, \eta, \xi) \in S^0(\mathbb{R}^{2n+1})$, such that $\chi = 1$ on $\Gamma = \{ (\tau, \eta, \xi) \in \mathbb{R}^{1+2n}, \vert \tau \vert^2 + \vert \eta \vert^2 \leq 1 + \vert \xi \vert^2 / 2 \}$, and $\chi = 0$ outside of $\tilde{\Gamma} = \{ \vert \tau \vert^2 + \vert \eta \vert^2 \geq 1 + \vert \xi \vert^2 \}$. Denoting by $\chi(D)$ the pseudo-differential operator with symbol $\chi$, it is enough to prove

$$\| A(v, D_t, D_x) \chi(D) f \|_{L^2(\mathbb{R}^{2n+1})} \leq C \{ \| a_1^{1/2}(D_v) \chi(D) f \|_{L^2(\mathbb{R}^{2n+1})}^2 + \| X_0 \chi(D) f \|_{H^{-s'}(\mathbb{R}^{2n+1})}^2 + \| f \|_{L^2(\mathbb{R}^{2n+1})}^2 \}.$$ 

Since supp $\mathcal{F}_{t,x,v}(\chi(D) f) \subset \tilde{\Gamma}$, one deduces that

$$\int \left( \int \frac{|\tau + v \cdot \eta|^2}{(1 + \tau^2 + |\eta|^2)^{n}} |\mathcal{F}_{t,x}(\chi(D) f)(v, \tau, \eta)|^2 \, d\tau \, d\eta \right) \, dv \leq \| X_0 \chi(D) f \|_{L^2(\mathbb{R}^{2n+1})}^2.$$ 

The operator $a_1^{1/2}(D_v)$ is independent of $t$, $x$, then it is enough to prove

$$c(\tilde{a}^- (v, \tau, \eta) u, u)_{L^2(\mathbb{R}^n)} \leq \text{Re}(\{ a_1(D_v) + a_2^+(v, \tau, \eta) \} u, u), \quad u \in S(\mathbb{R}^n), \quad (4)$$

for all $(\tau, \eta) \in \mathbb{R}^{1+n}$ such that $|\tau|, |\eta| \geq R_0$ for some big constant $R_0 > 0$, where for some $c_0 > 0$

$$a_1^-(v, \tau, \eta) = c_0 \frac{|\tau|^{2s/(1-s')/(s+1)}}{(1 + |\eta|^2)^{s/(s+1)}} + c_0 \frac{|\eta|^{2s/(1-s')/(s+1)}}{(1 + |\eta|^2)^{s/(s+1)}} = a_1^-(v, \tau) + a_2^-(v, \eta)$$

and

$$a_2^+(v, \tau, \eta) = 1 + \frac{|\tau + v \cdot \eta|^2}{(1 + \tau^2 + |\eta|^2)^{s'}}.$$ 

The final claim is that the condition (H) for the uncertainty principle in the form of (4) is true when $(\tau, \eta) \in \mathbb{R}^{1+n}$ is viewed as a parameter.

Finally, for Theorems 1.3 and 1.4, the existence and uniqueness of weak solution are obtained by approximation of equations with angular cutoff, and the regularity of weak solution is an application of Theorem 1.2 by using induction.

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References