ON THE INVERSE OF A CLASS OF DEGENERATE ELLIPTIC OPERATORS

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In this paper, the authors use the theory of nonhomogeneous symbolic calculus of pseudo-differential operators to study the fundamental solutions of a class of degenerate elliptic operators of type $D_{x_1}^2 + x_1^2 D_{x_2}^2$. From the results of Rothschild-Stein, those fundamental solutions are in the class of singular integral operators. The authors have improved those results in this paper, and proved that the fundamental solutions are in fact being pseudo-differential operators of Hörmander's class. So it is proved that, in the convenable weight Sobolev spaces and nonhomogeneous symbols class, the degenerate elliptic operators which we study in this paper is "elliptic."

Keywords Degenerate elliptic operator, Nonhomogeneous calculus, Microlocal analysis.
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§1. INTRODUCTION

In [2, 8], the authors studied the fundamental solutions for a class of Hörmander's operators. Using the theory of nonhomogeneous symbolic calculus developed in [1],[3],[5], they proved that...
the fundamental solution for a class of Hörmander operators is in the class of nonhomogeneous pseudo-differential operators. We study also this problem in this work, and consider in $\mathbb{R}^2$ the operators

$$P(x, D) = D_{x_1}^2 + \hat{\xi}_1^2 D_{x_2}^2 + c,$$

with an integer $k \geq 1$, $c > 0$, $\hat{\xi}_1 \in C^\infty(\mathbb{R}^1)$, $\hat{\xi}_1 = x_1$ for $|x_1| \leq 2$; $\hat{\xi}_1 = c|\text{sgn}x_1|$ for $|x_1| \geq 4$, where $c$ is a constant. Then $P(x, D)$ is a degenerate elliptic operator which satisfies the Hörmander’s condition of order $(k + 1)$. It is well known that the operators $P(x, D)$ are subelliptic (See [6]). Its parametrix does exist, and belongs to the pseudo-differential operators class of $S^{-2\delta}$ with $\delta = 1/(k+1)$.

In this work, we prove that the fundamental solution of operator (1.1) is a pseudo-differential operator. We now define the so-called Hörmander’s metrics and the admissible weight functions for operator of (1.1) (see Section 2). Set

$$M(x, \xi) = (\xi_1^2 + \hat{\xi}_1^2 \xi_2^2 + (\xi)^{2\delta})^{1/2},$$

$$G_{\xi, \xi}(dx, d\xi) = M^{-2/k}(x, \xi)(\xi)^{2/k}dx_1^2 + dx_2^2$$

$$+ M^{-2/k}(x, \xi)(\xi)^{-2\delta(1-1/k)}d\xi_1^2 + (\xi)^{-2}d\xi_2^2,$$

where $(\xi) = (1 + \xi_1^2 + \xi_2^2)^{1/2}, \delta = 1/(1+k)$. We can define as in [5] the nonhomogeneous symbol spaces $S(M^+, G)$, and the weight Sobolev spaces $H(M^+, G)$. If in (1.2), (1.3), $k = 0$, then we obtain the classical spaces $S^0_1(\mathbb{R}^2), H^1(\mathbb{R}^2)$. The main result of this paper is:

**Theorem 1.1.** In the above hypothesis, we have $P(x, D) \in Op(S(M^2, G))$, and there exists $P'(x, D) \in Op(S(M^{-2}, G))$ such that $P(x, D) \circ P'(x, D) = P'(x, D) \circ P(x, D) = I$, if $c$ is large enough.

We have proved that the fundamental solution of $P(x, D)$ is in the class of $S(M^{-2}, G)$. For all $s \in \mathfrak{N}$, if $f \in H(M^+, G)$, and $u \in S'(\mathbb{R}^2)$ is a solution of equation $P(x, D)u = f$, then $u \in H(M^{s+2}, G)$. This result means that, in this function spaces, the operators $P(x, D)$ have precise regularities properties as elliptic operators in the classical Sobolev spaces, and $P(x, D)$ is invertible in the algebra of pseudo-differential operators class $Op(S(M^+, G))$.

We will introduce some fundamental notations and results for nonhomogeneous pseudo-differential operators in section 2, study the weight Sobolev spaces and the microlocal analysis theory in section 3, and finally we prove the main Theorem 1.1 in section 4.

§2 HÖRMANDER SYMBOL CLASS AND WEYL CALCULUS

We give some results of [1, 5] here. For $a \in S'(\mathbb{R}^n)$, we define their Weyl calculus by

$$a^w(x, D)u(x) = (2\pi)^{-n} \int \int e^{i(x-y, \xi)}a(\frac{x+y}{2}, \xi)u(y)dyd\xi$$

$$= e^{\frac{i}{2}(D_yD_x)}a(\frac{y+z}{2})b(z)dYdZ$$

Then $a^w(x, D) : S(\mathbb{R}^n) \rightarrow S'(\mathbb{R}^n)$ is a continuous map. If $a, b \in S(\mathbb{R}^n)$, we define the composition of symbols by $(a \# b)^w(x, D) = a^w(x, D) \circ b^w(x, D)$, then we have for $X = (x, \xi) \in \mathbb{R}^n$,

$$a \# b(X) = \pi^{-2n} \int \int e^{2i\sigma(Y-X, Z-X)}a(Y)b(Z)dYdZ$$

$$= e^{\frac{i}{2}(D_yD_z)}a(\frac{Y+Z}{2})b(Z)|_{Y=Z=X},$$

(2.2)
with \( \sigma(Y, Z) = \sigma((y, \eta), (z, \zeta)) = (z, \eta) - (y, \zeta) \).

For \( X \in \mathbb{R}^{2n} \), let \( g_X(\cdot) \) be a positive definite form in \( \mathbb{R}^{2n} \). We call \( g \) continuous, if there exists a constant \( C_0 > 0 \) such that for all \( Y, Z, T \in \mathbb{R}^{2n} \),

\[
g_Z(Y) \leq C_0^{-1} \Rightarrow C_0^{-1} g_Z(T) \leq g_{Z+Y}(T) \leq C_0 g_Z(T)
\]

holds, and we call \( C_0 \) the continuous coefficient. For \( X, T \in \mathbb{R}^{2n} \), define

\[
g_X^T(T) = \sup_{W \neq 0} \frac{\sigma(T, W)^2}{g_X(W)}, \tag{2.4}
\]

\[
\lambda_X(T) = \inf_{T \neq 0} \left( \frac{g_X^T(T)}{g_X(T)} \right)^{1/2}. \tag{2.5}
\]

If for all \( X \in \mathbb{R}^{2n} \), \( \lambda_X(X) \geq 1 \), we say that \( g \) satisfies the uncertainty principle. \( g \) is temperate, if there exist the constants \( C_1, N_0 \) such that for all \( T \in \mathbb{R}^{2n} \setminus \{0\} \),

\[
\left( \frac{g_Y(T)}{g_X(T)} \right)^\pm \leq C_1 (1 + g_Y(Y-X))^{N_0}. \tag{2.6}
\]

In this case, the metric \( g \) is called Hormander's metric. A positive function \( m(X) \) \( (X \in \mathbb{R}^{2n}) \) is \( g \)-continuous, if there exists a constant \( C' \), such that for all \( X, Y \in \mathbb{R}^{2n} \),

\[
g_X(X-Y) \leq C'^{-1} \Rightarrow C'^1 \leq \frac{m(Y)}{m(X)} \leq C'. \tag{2.7}
\]

If there exist positive constants \( \overline{C}, N \) such that

\[
\left( \frac{m(Y)}{m(X)} \right)^\pm \leq \overline{C} (1 + g_Y(Y-X))^N, \tag{2.8}
\]

we call that \( m \) is a \( g \)-admissible weight function.

If \( g \) is continuous, and \( m \) is \( g \)-admissible weight function, we define the symbol space \( S(m, g) \) by

\[
S(m, g) = \{ a \in \mathcal{C}^\infty(\mathbb{R}^{2n}); ||a||_{k,m,g} < +\infty, \forall k \in \mathbb{N} \}, \tag{2.9}
\]

where the semi-norms is defined by

\[
||a||_{k,m,g} = \sup_{0 \leq |\xi| \leq \xi_0, X \in \mathbb{R}^{2n}} \{||(a^{(l)}(X), T_1 \otimes \cdots \otimes T_l)||m^{-1}(X)\}. \tag{2.10}
\]

Then we have \( S(m, g) \subset S'(\mathbb{R}^n) \). For a Hormander's metric, we define the \( g \)-ball by

\[
U_{X, r} = \{ Y \in \mathbb{R}^{2n}; g_X(X-Y) < r^2 \}. \tag{2.11}
\]

This "ball" is the base of nonhomogeneous microlocal analysis. We give the so-called "confine" notation.
Definition 2.1. A function \( a \in C^\infty(\mathbb{R}^{2n}) \) is confined on \( U_{Y,r} \), if for all integers \( k, N \), there exists constant \( C(k, N) > 0 \), such that for all \( X \in \mathbb{R}^{2n}, T_1, \cdots, T_k \in \mathbb{R}^{2n} \), we have

\[
|\langle a^{(k)}(X), T_1 \otimes \cdots \otimes T_k \rangle| 
\leq C(k, N) \prod_{j=1}^{k} g_X^{1/2}(T_j)(1 + g_Y(X - U_{Y,r}))^{-N/2},
\]

(2.12)

where \( g_Y(X - U_{Y,r}) = \inf_{Z \in U_{Y,r}} g_Y(X - Z) \).

In fact, the function spaces confined on \( U_{Y,r} \) is usual Schwartz space \( \mathcal{S}(\mathbb{R}^{2n}) \), but the important difference is the following "confine" semi-norms:

\[
\|a\|_{b, Y}^{U, Y, r} = \sup_{X, T_1, \cdots, T_k, k' \leq k} \left| \langle a^{(k')}(X), T_1 \otimes \cdots \otimes T_k \rangle \right| \times (1 + g_Y(X - U_{Y,r}))^{-N/2},
\]

(2.13)

\[
\|a\|_{b, p, Y}^{U, Y, r} = \|a\|_{b, Y}^{U, Y, r} = \|a\|_{b, Y}^{U, Y, r}.
\]

We need also the following symmetric distance function

\[
\Delta_r(X, Y) = 1 + \sup \{g_Y(U_{Y,r} - UX, r), g_X(U_{Y,r} - UX, r)\}.
\]

(2.15)

For the function \( \Delta_r(X, Y) \), there exist \( N_1, C_2 \) depend only on \( C_0, C_1, N_0 \) such that

\[
\int \Delta_r(X, Y)|g_Y|^{1/2}dY \leq C_2.
\]

(2.16)

If \( r^2 \leq C_0^{-1} \), there exist \( C_1, C' > 0 \), and \( N, \bar{N} \) such that for all \( X, Y, T \in \mathbb{R}^{2n} \setminus \{0\} \), and

\[
(g_X(T)/g_Y(T))^p \leq C_1' \Delta_1^N(X, Y),
\]

(2.17)

\[
(m(X)/m(Y))^p \leq C' \Delta_r^\bar{N}(X, Y)
\]

(2.18)

hold. In fact, for all \( X, Y, T \in \mathbb{R}^{2n} \setminus \{0\} \), we have

\[
g_X(T)/g_Y(T) \leq C_0^2 \inf_{X' \in U_{X,r}, Y' \in U_{Y,r}} (g_{X'}(T)/g_{Y'}(T)) \leq C_0^2 C_1 \inf_{X', Y'} (1 + g_Y(X') - Y'))^N \leq C_0^{N+2} C_1 (1 + g_Y(U_{X,r} - U_{Y,r}))^N \leq C_1 \Delta^N(X, Y).
\]

So

\[
(g_X(T)/g_Y(T))^p \leq C_1' \Delta_1^N(X, Y).
\]

Similarly

\[
(m(X)/m(Y))^p \leq C' \Delta_r^\bar{N}(X, Y).
\]
Given a Hörmander’s metric, we can construct a continuous partition of unity. For any \( r \leq C_0^{-1/2} \), there exists a family of function \( \{ \varphi_Y \}_{Y \in \mathbb{R}^{2n}} \) which uniformly confine on \( U_{Y,r} \), and \( \text{supp}\varphi_Y \subset U_{Y,r} \), such that

\[
\int_{\mathbb{R}^{2n}} \varphi_Y(X)|g_Y|^{1/2}dY = 1, \quad \forall X \in \mathbb{R}^{2n}.
\]

(2.19)

For the composition of two confined symbols, we have the following biconfine result, which means that if there exist two symbols confined on two \( g \)-balls respectively, then the composition of those two symbols is confined at the same time on the two balls. The proof is in Theorem 3.2.1 of [1].

**Lemma 2.1.** If \( g \) is a Hörmander’s metric on \( \mathbb{R}^{2n} \), then for all non-negative integers \( p, N, \nu \), there exist \( q \in \mathcal{N}, C > 0 \), such that for all \( a, b \in S(\mathbb{R}^{2n}), r^2 \leq C_0 \), we have

\[
||a\# b(X) - \sum_{0 \leq j < \nu} \frac{1}{j!} \left( \frac{i}{2} \left( \sigma(D_{X_1}, D_{X_2}) \right) \right)^j (a \otimes b)(x_1 = x_2 = x) ||_{p, Y,r} \\
\leq C||a||_{q,Y,r} ||b||_{q,Z,r} \lambda_{y}^{-\nu}(Y) \Delta_{z}^{-N}(Y, Z).
\]

(2.20)

Using this results, we have

**Lemma 2.2.** Let \( g \) be a Hörmander’s metric, \( m_1, m_2 \) be two \( g \)-admissible weight functions. Then for any \( a \in S(m_1, g), b \in S(m_2, g) \), we have \( a\# b \in S(m_1 m_2, g) \). And for any \( k \in \mathcal{N}, \) there exist \( l \in \mathcal{N}, C > 0 \) such that

\[
||a\# b||_{k,S(m_1 m_2, g)} \leq C||a||_{l,S(m_1, g)} ||b||_{l,S(m_2, g)}.
\]

(2.21)

This result means that the symbol spaces form a “graduate” algebra defined by the multiplication of weight functions.

For \( a \in S(m, g) \), we define the operator norm by

\[
||a||_{k, Op(S(m, g))} = \sup_{p \leq k, Y, \varphi(T) \leq 1} ||(ad L_1) \circ \cdots \circ (ad L_p) \circ (\varphi^w \circ a^w)||_{L(L^2)}.
\]

(2.22)

where the operators \( L_j = \sigma(T_j, 1)^w, \) \( ad L \circ A = [L, A] \). It is evident that \( ad L \circ a^w = (\partial\varphi^T) a^w \).

And for any \( k \in \mathcal{N}, \) there exist \( C > 0, l \in \mathcal{N}, \) such that

\[
||a||_{k,S(m,g)} \leq C||a||_{l, Op(S(m,g))}.
\]

(2.23)

In order to study the inverse of a pseudo-differential operator, we assume that the metric \( g \) satisfies some additional hypothesis.

**Definition 2.2.** A metric \( g \) is strongly temperate, if there exists a function \( d(X, Y) > 0 \) defined on \( \mathbb{R}^{2n} \times \mathbb{R}^{2n} \), and there exist \( C > 0, N \geq 0, r > 0, \) such that for all \( X, Y, Z \in \mathbb{R}^{2n} \setminus \{0\}, \) we have

\[
d(X, Y) \leq d(X, Z) + d(Z, Y),
\]

(2.24)

\[
(g_X(T)/g_Y(T))^N \leq C(1 + d(X, Y))^N,
\]

(2.25)

\[
1 + d(X, Y) \leq C\Delta^N(X, Y).
\]

(2.26)
A metric $g$ can be dominated by a strongly temperate metric, if there exists a strongly temperate metric $\tilde{g}$, and $C > 0, N \geq 0$, such that for any $X, Y \in \mathbb{R}^{2n}, T \in \mathbb{R}^{2n} \setminus \{0\}$, we have

$$g_x \leq \tilde{g}_x,$$

$$(g_x(T)/\tilde{g}_x(T))^\pm \leq C(1 + \tilde{g}_x(X - Y))^N.$$  

**Example 2.1.** The metric for the classical pseudo-differential operators is $g^{x, \xi} = (\xi)^{2\varepsilon} dx^2 + (\xi)^{-2\varepsilon} d\xi^2$. If $0 \leq \delta \leq \rho < 1$, this metric is strongly temperate. In fact, for $X = (x, \xi), Y = (y, \eta)$, we may take $d(X, Y) = |(\xi)^{1-\rho} - (\eta)^{1-\rho}|$, and (2.24)-(2.26) as evident. If $0 \leq \delta \leq \rho \leq 1, \delta < 1$, the metric $g^{x, \xi}$ can be dominated by the strongly temperate metric $g^{x, \xi}$.

Now we use the continuous partition of unity (2.19) to define the weight of Sobolev spaces. Let $g$ be a strongly temperate metric, or able to be dominated by a strongly temperate metric. Let $m$ be an admissible weight function. Then define the associated Sobolev space $H(m, g)$ by

$$H(m, g) = \{u \in S'(\mathbb{R}^n); \int m^2(Y)\|\varphi \psi u\|_{L^2}^2 |g_Y|^{1/2} dY < +\infty\} \quad (2.29)$$

with norm

$$\|u\|_{H(m, g)}^2 = \int m^2(Y)\|\varphi \psi u\|_{L^2}^2 |g_Y|^{1/2} dY. \quad (2.30)$$

Then $H(m, g)$ is a Hilbert space. If $m, m_1$ are two $g$-admissible weight functions, then for any $a \in S(m, g)$, we have

$$a^w : H(m_1, g) \to H(m_1/m, g), \quad (2.31)$$

and

$$H(1, g) = L^2(\mathbb{R}^2). \quad (2.32)$$

On the other hand, if $g$ is strongly temperate or it can be dominated by a strongly temperate metric, then we can construct a double partition of unity, i.e. if $\{\varphi_Y\}$ is a $g$-partition of unity defined by (2.19), supp$\varphi_Y \subset U_{Y,r}$, then there exists a family $\{\psi_Y\}$ uniformly confined in $U_{Y,r}$, such that

$$\int \psi_Y \# \varphi_Y |g_Y|^{1/2} dY = 1. \quad (2.33)$$

The following is a theorem of J. M. Bony and J. Y. Chemin.[3] By this theorem, when we want to prove if the inverse of a pseudo-differential operator is also a pseudo-differential operator, we need only prove that this operator is a bijection between some weight Sobolev spaces.

**Theorem 2.1.** Assume that $g$ is a strongly temperate Hörmander's metric or that it can be dominated by a strongly temperate metric, $m$ is a $g$-admissible weight function, $a \in S(m, g)$. If there exists an admissible weight function $m_1$ such that $a^w : H(m_1, g) \to H(m_1/m, g)$ is a bijection, then there exists $a' \in S(m^{-1}, g)$, such that $a' \# a = a \# a' = 1$.

§3 HÖRMANDER'S METRIC AND WEIGHTS SOBOLEV SPACES

We now study the operator $P(x, D)$ introduced in section 1; the metric $G$ and weight function $M$ are defined in (1.3), (1.2).

**Lemma 3.1.** The metric $G$ is continuous, and $M$ is a $G$-admissible weight function.
Proof. Set $\tilde{C} = \sup |\tilde{z}|$, then $\tilde{C} \geq 2$. To prove that $G$ is continuous, we have to prove that there exist $c, C > 0$ such that for any $X, Y, T \in \mathbb{R}^4$, we have

$$G_{X}(Y) \leq c \Rightarrow G_{X+Y}(T) \leq CG_{X}(T).$$

In fact, we have $M(X) = (\xi_1^2 + \xi_2^2 + \langle \xi \rangle^{2k})^{1/2} \leq (\tilde{C}^{2k} + 1)^{1/2}(\xi)$, then

$$M^{-2/k}(X)(\xi)^{-2k(1-1/k)} \geq (\tilde{C}^{2k} + 1)^{-1/k}(\xi)^{-2k(1-1/k)} \geq (\tilde{C}^{2k} + 1)^{-1/k}(\xi)^{-2}.$$

Take $c \leq 1/4(\tilde{C}^{2k} + 1)^{-1/k}$, then $G_{X}(Y) \leq c$ gives $|\eta|^2 \leq 1/4(\xi)^2$, so that

$$(\xi)^2/8 \leq (\xi + \eta)^2 \leq 8(\xi)^2.$$ 

Then $G$ is continuous, only if there exists $C'$ such that

$$G_{X}(Y) \leq c \Rightarrow M(X) \leq C'M(X + Y).$$

But from $G_{X}(Y) \leq c$ we have

$$M^{-2/k}(X)(\xi)^{2/k}y^2 \leq c \Rightarrow |y_1| \leq 1/2.$$

If $|x_1| \geq 1$,

$$M(X + Y) = \{((\xi_1 + \eta_1)^2 + (\xi_2 + \eta_2)^2 + (\xi + \eta)^{2k})^{1/2} \geq 2^{-k}(\xi + \eta).$$

Since $|x_1| \geq 1$, for $C' \geq 2^{2k+3}(\tilde{C}^{2k} + 1)$, the condition $G_{X}(Y) \leq c$ gives

$$M(X) \leq C'M(X + Y).$$

If $|x_1| \leq 1$, we divide it into three cases:

(a) When $(\xi)^6 \geq 1/3M(X)$, take $c \leq 1/4(\tilde{C}^{2k} + 1)^{-1/k}$, the condition $G_{X}(Y) \leq c$ gives

$$M(X) \leq 3(\xi)^6 \leq 24(\xi + \eta)^6 \leq 24M(X + Y).$$

(b) When $|\xi_1| \geq 1/3M(X)$, take $c \leq 1/12$, the condition $G_{X}(Y) \leq c$ gives

$$\eta_1^2 \leq c(\xi)^{2k(1-1/k)}M^{3/k}(X) \leq cM^2(X) \leq 1/4\xi^2,$$

so that $(\xi_1 + \eta_1)^2 \geq 1/4\xi^2$, and

$$M(X) \leq 3|\xi_1| \leq 12|\xi_1 + \eta_1| \leq 12M(X + Y).$$

(c) When $x_1^2\xi_2^2 \geq 1/3M^2(X)$, take $c \leq 1/48$, then the condition $G_{X}(Y) \leq c$ gives

$$\eta_1^2 \leq c(\xi)^2 \leq 3c\xi_2^2 \leq 1/16\xi_2^2.$$
Hence \((\xi_2 + \eta_2)^2 \geq 1/4\xi_2^2\), and

\[
M^{-2/k}(X)(\xi)^{2/k} y_1^2 \leq 2 \Rightarrow |y_1| \leq c^{1/2}(3x_1^k \xi_2^2)/(\xi)^{2/(2k)} \leq 1/4|x_1|,
\]

which gives

\[
M^2(X) \leq 3x_1^k \xi_2^2 \\
\leq 3 \times 2^{k}(x_1 + y_1)^{2k} \times 4(\xi_2 + \eta_2)^2 \\
\leq 2^{2k+4} M^2(X + Y).
\]

We have finally, if \(c \leq \min \{1/48, 1/4(\tilde{C}^{2k} + 1)^{-1/k}\}, C' \geq 2^{2k+3}(\tilde{C}^{2k} + 1)\), then

\[
G_X(Y) \leq c \Rightarrow M(X) \leq C'M(X + Y).
\]

And for \(c \leq 1/12(\tilde{C}^{2k} + 1)^{-1/k}, C \geq 2^{\delta}(\tilde{C}^{2k} + 1)^{1/k}\), we have obtained

\[
G_X(Y) \leq c \Rightarrow G_{X+Y}(T) \leq CG_X(T).
\]

We have proved that \(G\) is continuous, and at the same time \(M\) is a \(G\)-admissible weight function.

Since \(G\) is diagonal, we have from the definition

\[
\begin{align*}
G^\mathbf{x}(dx, d\xi) &= M^{2/k}(X)(\xi)^{2k} dx_1^2 + (\xi)^{2} dx_2^2 \\
&+ M^{2/k}(X)(\xi)^{-2/k} d\xi_1 + d\xi_2. 
\end{align*}
\]

(3.1)

For the metric \(G\) and weight function \(M\), we have

**Lemma 3.2.** The metric \(G\) can be dominated by the strongly temperate metric \(g^{\delta, \delta}\), \(M\) is a \(G\)-admissible weight function.

**Proof.** For

\[
g^{\delta, \delta}(dx, d\xi) = (\xi)^{2\delta} dx^2 + (\xi)^{-2\delta} d\xi^2,
\]

we have evident \(G_X \leq g^{\delta, \delta}_X \leq G_X^\mathbf{x}\). We now prove

\[
\begin{align*}
\left(\frac{G_X(\cdot)}{G_Y(\cdot)}\right)^{\pm 1} &\leq C_1(1 + g^{\delta, \delta}_Y(X - Y))^{N}, \\
\left(\frac{M(X)}{M(Y)}\right)^{\pm 1} &\leq \tilde{C}(1 + g^{\delta, \delta}_Y(X - Y))^{N}.
\end{align*}
\]

(3.3)

For (3.4), since

\[
\frac{(\xi)^2}{\eta^2} \leq \begin{cases}
4, & \text{if } (\xi) \leq 2(\eta), \\
\frac{4(1 + (\xi - \eta)^2)}{(\eta)^2}, & \text{if } (\xi) > 2(\eta),
\end{cases}
\]

\[
\frac{(\eta)^2}{(\xi)^2} \leq \begin{cases}
4, & \text{if } (\eta) \leq 2(\xi), \\
\frac{(\eta)^2}{(\xi)^2} \leq \left(\frac{\xi}{\eta}\right)^{2k} \leq 16(1 + g^{\delta, \delta}_Y(X - Y))^2, & \text{if } (\eta) > 2(\xi),
\end{cases}
\]

\[
\frac{\eta^2}{M^2(X)} \leq \begin{cases}
4\frac{\eta^2}{(\eta)^{2k}(\xi)^{2k}} \leq \frac{4(1 + g^{\delta, \delta}_Y(X - Y))}{(\eta)^{2k}(\xi)^{2k}} \leq 16(1 + g^{\delta, \delta}_Y(X - Y))^2, & \text{if } |\eta| \leq 2|\xi|,
\end{cases}
\]

\[\text{if } |\eta| > 2|\xi|.
\]
On the other hand, if \(|y_1| > 2|x_1|\), we have
\[
\frac{y_1^{2k} \eta_2^2}{M^2(X)} \leq y_1^{2k}(\eta)^2(\xi)^{-26} \leq 2^{2k}|y_1 - x_1|^{2k}(\eta)^{2-26} \\
\leq 2^{2k+2}(1 + g_{\bar{\nu}}^{\xi}(X - Y))^k+1.
\]

If \(2|\xi| < |\eta|\), we have
\[
\frac{y_1^{2k} \eta_2^2}{M^2(X)} \leq 4\tilde{C}^{2k} \frac{|\xi - \eta|}{(\eta)^{26}}(\xi)^{26} \\
\leq 16\tilde{C}^{2k}(1 + g_{\bar{\nu}}^{\xi}(X - Y))^2,
\]
and if \(|\eta| \leq 2|x_2|, |y_1| \leq 2|x_1|\), then
\[
\frac{y_1^{2k} \eta_2^2}{M^2(X)} \leq 4 \max\{2^{2k}, 2^{-2k}\tilde{C}^{2k}\}.
\]

Thus we have
\[
\frac{M(Y)}{M(X)} \leq 4(\tilde{C}^{2k} + 2)^{1/2}(1 + g_{\bar{\nu}}^{\xi}(X - Y))^{(k+1)/2}.
\]

In a similar way, we also obtain
\[
\frac{M(X)}{M(Y)} \leq 4(\tilde{C}^{2k} + 2)^{1/2}(1 + g_{\bar{\nu}}^{\xi}(X - Y))^{(k+1)/2}.
\]

Hence, for \(\tilde{C} = 4(\tilde{C}^{2k} + 2)^{1/2}, N = (k + 1)/2\), we have
\[
(M(X)/M(Y))^N \leq \tilde{C}(1 + g_{\bar{\nu}}^{\xi}(X - Y))^N.
\]

From the definition of metric \(G\), for \(C_1 = (16\tilde{C})^{1/k}, N = 3\), we have
\[
(Gx(\cdot)/Gy(\cdot))^N \leq C_1(1 + g_{\bar{\nu}}^{\xi}(X - Y))^N.
\]

We have proved the lemma.

Denote by \(p(x, \xi)\) the symbol of \(P(x, D)\), then \(p(x, \xi) = \xi_t^2 + z_1^{2k}\xi_2^2 + c\), and

**Lemma 3.3.** With above notations, we have \(M^2, p \in S(M^2, G)\).

**Proof.** We prove first \(p \in S(M^2, G)\), and divide it into three cases:
(a) differentiate with respect to \(x_1\): if \(l \leq 2k\), we have
\[
\partial_{x_1}^l p(x, \xi) = \xi_t^2[2k(2k - 1) \cdots (2k - l + 1)z_1^{2k-l}(z_1')^l] \\
+ \sum_{j=1}^{l-1} z_1^{2k-j} \partial_{x_1}^l - j(z_1')^l,
\]
and if \(|x_1| \leq 1\), we obtain
\[
\partial_{x_1}^l p = \xi_t^2[2k(2k - 1) \cdots (2k - l + 1)z_1^{2k-l}],
\]
which gives

\[
\frac{|\partial_{x_1}^l p|}{M^{2-1/k}(X)} \leq \frac{2k(2k-1)\cdots(2k-l+1)|x_1^{2k-l}\xi_2^2|}{(\xi_1^2 + x_1^{2k}\xi_2^2 + (\xi)^2)^{1-1/2k}} \\
\leq C_{1,l}(\xi)^{1/k}.
\]

If \(|x_1| > 1\), we have

\[
|\partial_{x_1}^l p| \leq C_{1,l}\xi_2^2 \leq C_{1,l}M^2(X) \leq C_{1,l}M^2(X)G^{1/2}_X(1,0,0,0).
\]

Consider now \(l > 2k\). If \(|x_1| \leq 1\), we have \(\partial_{x_1}^l p(X) = 0\); if \(|x_1| > 1\) we have immediately

\[
|\partial_{x_1}^l p(X)| \leq C_{1,l}M^2(X)G^{1/2}_X(1,0,0,0).
\]

So we have proved that for any \(l \in \mathcal{N}\), there exists \(\tilde{C}_{1,l}\), such that

\[
|\partial_{x_1}^l p(X)| \leq \tilde{C}_{1,l}M^2(X)G^{1/2}_X(1,0,0,0).
\]

(b) differentiate with respect to \(\xi_1\):

\[
|\partial_{\xi_1}^l p(X)|M^{-1/2}(X) \leq 2|\xi_1|(\xi_1^2 + (\xi)^2)^{-(1-1/2k)} \\
\leq 2(|\xi_1|^{(2k-2)/(2k-1)} + (\xi)^{2d}|\xi_1|^{-(2k-2)/2k}-(2k-1)/2k \\
\leq C_{3,1}(\xi)^{-(2k(2k-2)/(2k-1))} \leq C_{3,1}(\xi)^{-\delta(1-1/k)}.
\]

Hence

\[
|\partial_{\xi_1}^l p(X)| \leq C_{3,1}M^2(X)G^{1/2}_X(0,0,1,0), \\
|\partial_{\xi_1}^l p(X)| \leq 2M^2(X)G_X(0,0,1,0).
\]

If \(l > 2\), \(\partial_{\xi_1}^l p(X) = 0\).

(c) differentiate with respect to \(\xi_2\):

\[
|\partial_{\xi_2}^l p(X)|M^{-2}(X) \leq 2\xi_2^2k|\xi_2|M^{-2}(X) \\
\leq 2\xi_2^{2k}|\xi_2|\xi_2^{\delta} \\
\leq C_{4,1}(\xi)^{-1} \leq C_{4,1}G^{1/2}_X(0,0,0,1), \\
|\partial_{\xi_2}^l p(X)|M^{-2}(X) \leq C_{4,2}(\xi)^{-2} \leq C_{4,2}G^{1/2}_X(0,0,1,0).
\]

If \(l > 2\), \(\partial_{\xi_2}^l p(X) = 0\).

Combine the results of (a) and (c), for any \(l_1, l_2 \in \mathcal{N}\) there exists \(C_{l_1+l_2} > 0\) such that

\[
|\partial_{x_1}^l \partial_{\xi_2}^m p(X)| \leq C_{l_1+l_2}M^2(X)G^{1/2}_X(1,0,0,0)G^{1/2}_X(0,0,0,1).
\]
So that for any \( l \in \mathcal{N} \) there exists \( C_l > 0 \), such that for all \( T_j \in \mathbb{R}^4, j = 1, \ldots, l \), we have

\[
|\langle \phi^{(l)}(X), T_1 \otimes \cdots \otimes T_l \rangle| \leq C_l M^2(X) \prod_{j=1}^l G_{X}^{1/2}(T_j),
\]

which proves \( p \in S(M^2, G) \).

We prove now \( \langle \xi \rangle^{26} \in S(M^2, G) \),

\[
|\xi_1^{(l)}(\xi_2)^{26}| = \sum_{l_1-2p \geq 0} \sum_{l_2 \geq 0} C_{l_1,p} C_{l_2,q} \xi_1^{(l_1-2p)} \xi_2^{(l_2-q)} \langle \xi \rangle^{26}. 
\]

On the other hand

\[
|M^{-2/k}(X)(\xi)^{-26(1-1/k)\xi^{1/2}} \geq (C^{2k} + 1)^{-1/2k} \langle \xi \rangle^{-26}. 
\]

Therefore for any \( l_1, l_2 \geq 0 \), there exists \( C_{l_1,l_2} > 0 \), such that

\[
|\xi_1^{(l_1)}(\xi_2)^{26}| \leq C_{l_1,l_2} M^2(X)G_{X}^{1/2}(0, 0, 0, 1). 
\]

This proves that \( \langle \xi \rangle^{26} \in S(M^2, G) \), so \( M^2 \in S(M^2, G) \). We have proved the lemma.

§4 THE PROOF OF THEOREM 1.1

Using Theorem 2.1, we need only to prove that for some \( k \in \mathbb{N} \), \( P(x, D) : H(M^k, G) \rightarrow H(M^{k-2}, G) \) is a bijection. We set

\[
S = \{(y, \eta) \in \mathbb{R}^4; M^2(y, \eta) \leq A^k(\eta)^{26}\},
\]

\[
E = \mathbb{R}^4 \setminus S,
\]

where \( A \) is a big constant to be fixed. If \( A \geq 4(C^{2k} + 1) \), for any \( Y \in E \), we have

\[
\lambda_{G}(Y) \geq A. 
\]

Lemma 4.1. Let \( \{\varphi_Y\} \) be a continuous \( G \)-partition of unity. Then there exist two families of symbols \( \{\delta_Y\}, \{R_Y\} \) which are uniformly confined on \( U_{Y,r} \), and

\[
\varphi_Y(X) = M^{-2}(Y)\delta_Y \#M^2(X) + \lambda_{G}^{-1}R_Y(X), \ Y \in E. 
\]

Proof. Firstly

\[
\varphi_Y(X) = M^{-2}(Y)\frac{M^2(Y)}{M^2(X)}\varphi_Y(X) \int M^2(Z)M_Z(X)|G_Z|^{1/2}dZ, 
\]
where $M_Z(X) = C^{M^2(X)}$, $h$ is an integer large enough, such that $1 = \int C^{M^2-h}(Z)|G_Z|^{1/2} \, dZ$.

We set

$$\delta_Y(X) = \begin{cases} \frac{M(Y)}{M(X)} \varphi_Y(X), & Y \in E, \\ 0, & Y \in S. \end{cases}$$

which is uniformly confined on $U_{Y,r}$. Hence, if $Y \in E$, we have

$$\varphi_Y(X) = M^2(Y) \int \delta_Y(X) M_Z(X) M^2(Z)|G_Z|^{1/2} \, dZ,$$

$$M_Z(X) \delta_Y(X) = \delta_Y \# M_Z(X) + \lambda_0^{-1}(Y) R_Y(X) \Delta_{r-N}(Y, Z),$$

where \{ $R_Y$ \} is uniformly confined on $U_{Y,r}$. Then

$$\varphi_Y(X) = M^{-2}(Y) \int M^2(Z) \delta_Y \# M_Z(X)|G_Z|^{1/2} \, dZ$$

$$+ \lambda_0^{-1}(Y) \int \frac{M^2(Z)}{M^2(Y)} \Delta_{r-N}(Y, Z) R_Y|G_Z|^{1/2}.$$

Take $N \geq N_1 + N$, and

$$R_Y = R_Y \int \frac{M^2(Z)}{M^2(Y)} \Delta_{r-N}(Y, Z)|G_Z|^{1/2} \, dZ.$$

Then, \{ $R_Y$ \} is uniformly confined on $U_{Y,r}$. So we have proved that for any $Y \in E$,

$$\varphi_Y(X) = M^{-1}(Y) \delta_Y \# M(X) + \lambda_0^{-1}(Y) R_Y(X).$$

Denote now $Q(z, D) = D_z^2 + \hat{x}_1^2 \hat{x}_2^2$, $q(z, \xi) = \xi_1^2 + \hat{x}_1^2 \hat{x}_2^2$, then $q(z, \xi) \in S^{0,0}$, $q(z, \xi) \geq 0$.

By sharp Garding inequality, there exists $C_1 > 0$, such that for any $u \in C_0^\infty(\mathbb{R}^2)$, we have

$$Re(Q(z, D)u, u) \geq -C_1 \|u\|_{L^2}^2.$$  \hspace{1cm} (4.3)

Since the operator $P(z, D)$ is subelliptic, if the coefficient $c$ is large enough, we have the subelliptic estimate

$$\|u\|_{H^{2c}} \leq C\|P(z, D)u\|_{L^2}, \quad \forall u \in C_0^\infty(\mathbb{R}^2).$$  \hspace{1cm} (4.4)

We prove now the main lemma.

**Lemma 4.2.** There exist constants $C_1, C_2 > 0$, such that for all $u \in H(M^2, G)$, we have

$$C_1\|u\|_{H(M^2, G)} \leq \|P(z, D)u\|_{L^2} \leq C_2\|u\|_{H(M^2, G)}.$$  \hspace{1cm} (4.5)

**Proof.** Firstly, since $p(z, \xi) \in S(M^2, G)$, there exists $C_2 > 0$ such that

$$\|P(z, D)u\|_{L^2} \leq C_2\|u\|_{H(M^2, G)}.$$
For the other side of (4.5), we have for any \( u \in C_0^\infty(\mathbb{R}^2) \),
\[
\|u\|_{H(M^2, G)}^2 = \int_E M^4(Y) \|\varphi^\omega u\|^2_{L^2} |G_Y|^{|1/2|} \, dY \\
= \int_S M^4(Y) \|\varphi^\omega u\|^2_{L^2} |G_Y|^{|1/2|} \, dy + \int_E m^4(y) \|\varphi^\omega y\|^2_{L^2} |g_y|^{|1/2|} \, dy \\
= 1 + II
\]
\[
I = \int_S M^4(Y) \|\varphi^\omega u\|^2_{L^2} |G_Y|^{|1/2|} \, dy \\
\leq A^{2k} \int_S ((\xi)^{2k})^2 \|\varphi^\omega u\|^2_{L^2} |G_Y|^{|1/2|} \, dy \\
\leq C_3 A^{2k} \|u\|^2_{L^2}.
\]
From the subelliptic estimates (4.4), we have
\[
I \leq C_3 C_4 A^{2k} \|P(x, D) u\|^2_{L^2}.
\]
For the estimate of II, we have
\[
II = \int_E M^4(Y) \|\varphi^\omega u\|^2_{L^2} |G_Y|^{|1/2|} \, dY \\
\leq 2 \int_E M^4(Y) \|M^{-2}(Y)(\delta Y \# M^2)^\omega u\|^2_{L^2} |G_Y|^{|1/2|} \, dY \\
+ 2 \int_E M^4(Y) \|\lambda_G^{-1}(Y) R_Y^\omega u\|^2_{L^2} |G_Y|^{|1/2|} \, dY \\
\leq 4 \int_E \|\delta_Y^\omega o (\omega(M^2) - M^2(x, D)) u\|^2_{L^2} |G_Y|^{|1/2|} \, dY \\
+ 4 \int_E \|M^2(x, D) u\|^2_{L^2} |G_Y|^{|1/2|} \, dY \\
+ 2 \int_E M^4(Y) \|\lambda_G^{-3}(Y) R_Y^\omega u\|^2_{L^2} |G_Y|^{|1/2|} \, dY \\
\leq 4 A^{-2} \int_E \lambda_G^3(Y) \|\delta_Y^\omega o (\omega(M^2) - M^2(x, D)) u\|^2_{L^2} |G_Y|^{|1/2|} \, dY \\
+ C_5 \|M^2(x, D) u\|^2_{L^2} + C_6 A^{-2} \|u\|^2_{H(M^2, G)} \\
\leq C_7 A^{-2} \|\omega(M^2) - M^2(x, D)) u\|^2_{H(L, \lambda, G)} \\
+ 2 C_5 \|Q(x, D) u\|^2_{L^2} + 2 C_5 \|u\|^2_{L^2} + C_6 A^{-2} \|u\|^2_{H(M^2, G)}.
\]
Hence
\[
II \leq (C_6 + C_7 \cdot C_8) A^{-2} \|u\|^2_{H(M^2, G)} + 2 C_5 (1 + C_4) \|P(x, D) u\|^2_{L^2}.
\]
Take \( A \geq \max\{4(C_3^2 + 1), 2(C_7 C_8 + C_6)^{1/2}\} \), we obtain
\[
\|u\|^2_{H(M^2, G)} \leq 2(C_3 C_4 A^{2k} + 2(C_5 + C_6 C_4)) \|P(x, D) u\|^2_{L^2}.
\]
Then take \( C_1 \leq 2(c_3 c_4 a^{2k} + 2(c_5 + c_5 c_4))^{-1/2} \), we have
\[
C_1 \|u\|_{H(M^2, G)} \leq \|P(x, D) u\|_{L^2}.
\]
Since $C_0^\infty(\mathbb{R}^2)$ is dense in $H(M^2, G)$, we can also obtain this estimate for any $u \in H(M^2, G)$, which proves the lemma.

We have finally proved that if $c$ is big enough, $P(x, D) : H(M^2, G) \to L^2(\mathbb{R}^2)$ is a bijection. Using Theorem 2.1, we have proved the main results of this paper, Theorem 1.1.

REFERENCES