Precise formulas for the distributions of the principal geometric characteristics of the typical cells of a two-dimensional Poisson-Voronoi tessellation and a Poisson line process. *

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Abstract

In this paper, we give an explicit integral expression for the joint distribution of the number and the respective positions of the sides of the typical cell $C$ of a two-dimensional Poisson-Voronoi tessellation. We deduce from it precise formulas for the distributions of the principal geometric characteristics of $C$ (area, perimeter, area of the fundamental domain). We also adapt the method to the Crofton cell and the empirical (or typical) cell of a Poisson line process.

1 Introduction and principal results.

1.1 The typical cell of a two-dimensional Poisson-Voronoi tessellation.

Consider $\Phi$ a homogeneous Poisson point process in $\mathbb{R}^2$, with the two-dimensional Lebesgue measure $V_2$ for intensity measure. The set of cells

$$ C(x) = \{ y \in \mathbb{R}^2; ||y - x|| \leq ||y - x'||, x' \in \Phi \}, \quad x \in \Phi, $$

(which are almost surely bounded polygons) is the well-known Poisson-Voronoi tessellation of $\mathbb{R}^2$. Introduced by Meijering [13] and Gilbert [6] as a model of crystal aggregates, it provides now models for many natural phenomena such as thermal conductivity [9], telecommunications [3], astrophysics [22] and ecology [19]. An extensive list of the areas in which the tessellation has been used can be found in Stoyan et al. [20] and Okabe et al. [18].

In order to describe the statistical properties of the tessellation, the notion of typical cell $C$ in the Palm sense is commonly used [17]. Consider the space $\mathcal{K}$ of convex compact sets of $\mathbb{R}^2$ endowed with the usual Hausdorff metric. Let us fix an arbitrary Borel set $B \subset \mathbb{R}^2$ such that $0 < V_2(B) < +\infty$. The typical cell $C$ is defined by means of the identity [17]:

$$ \mathbb{E} h(C) = \frac{1}{V_2(B)} \mathbb{E} \sum_{x \in B \cap \Phi} h(C(x) - x), $$

where $h : \mathcal{K} \rightarrow \mathbb{R}$ runs throughout the space of bounded measurable functions.

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Consider now the cell
\[ C(0) = \{ y \in \mathbb{R}^2; ||y|| \leq ||y - x||, x \in \Phi \} \]

obtained when the origin is added to the point process \( \Phi \). It is well known [17] that \( C(0) \) and \( C \) are equal in law. From now on, we will use \( C(0) \) as a realization of the typical cell \( C \). We will call a point \( y \) of \( \Phi \) a neighbor of the origin if the bisecting line of the segment \( [0, y] \) intersects the boundary of \( C(0) \). Let us denote by \( N_0(C) \) the number of sides (or equivalently vertices) of the typical cell \( C \). In [5], we provided an integral formula for the distribution function of \( N_0(C) \). We extend the method to obtain the joint distribution of the respective positions of the \( k \) lines bounding \( C(0) \) conditionally to the event \( \{ N_0(C(0)) = k \} \), \( k \geq 3 \).

**Theorem 1** (i) For every \( k \geq 3 \), we have
\[
\mathbb{P}\{ N_0(C) = k \} = \frac{(2\pi)^k}{k!} \int d\sigma_k(\delta_1, \ldots, \delta_k) \prod_{i=1}^k e^{-H(\delta_i, p_i, p_{i+1})} 1_B(p_{i-1}, p_i, p_{i+1}, \delta_{i-1}, \delta_i) dp_i, \tag{1}
\]
where \( \sigma_k \) is the (normalized) uniform measure on the simplex
\[
S_k = \{ (\delta_1, \ldots, \delta_k) \in [0, 2\pi]^k; \sum_{i=1}^k \delta_i = 2\pi \}, \tag{2}
\]
with
\[
B = \{ (p, q, r, \alpha, \beta) \in (\mathbb{R}_+)^3 \times (0, \pi)^2; p \sin(\beta) + r \sin(\alpha) \geq q \sin(\alpha + \beta) \}, \tag{3}
\]
with for every \( \delta \in (0, \pi) \), \( p, q \geq 0 \),
\[
H(\delta, p, q) = \frac{1}{2\sin^2(\delta)} \left\{ (p^2 + q^2 - 2pq \cos(\delta)) + pq \sin(\delta) - \frac{p^2}{4} \sin(2\delta) - \frac{q^2}{4} \sin(2\delta) \right\}, \tag{4}
\]
and with the conventions \( p_0 = p_k, p_{k+1} = p_1 \), and \( \delta_0 = \delta_k \);
(ii) conditionally to \( \{ N_0(C(0)) = k \} \), let us denote by \( (P_1, \Theta_1), \ldots, (P_k, \Theta_k) \) the polar coordinates of the consecutive neighbors of the origin in the trigonometric order.

The joint distribution of the vector \( (P_1, \ldots, P_k, \Theta_2 - \Theta_1, \ldots, \Theta_k - \Theta_{k-1}, 2\pi + \Theta_1 - \Theta_k) \)
then has a density with respect to the measure
\[
d\nu_k(p_1, \ldots, p_k, \delta_1, \ldots, \delta_k) = dp_1 \cdots dp_k d\sigma_k(\delta_1, \ldots, \delta_k), \tag{5}
\]
and its density \( \varphi_k \) is given by the following equality for every \( p_1, \ldots, p_k \geq 0 \), \( (\delta_1, \ldots, \delta_k) \in S_k \),
\[
\varphi_k(p_1, \ldots, p_k, \delta_1, \ldots, \delta_k) = \frac{1}{\mathbb{P}\{ N_0(C) = k \} k!} \prod_{i=1}^k p_i e^{-H(\delta_i, p_i, p_{i+1})} 1_B(p_{i-1}, p_i, p_{i+1}, \delta_{i-1}, \delta_i).
\]
A table of numerical values for the distribution function of \( N_0(C) \) has already been provided (see [5], table 1).

Let us denote by \( \mathcal{F}(C(0)) \) the fundamental domain associated to \( C(0) \), i.e.
\[
\mathcal{F}(C(0)) = \cup_{x \in C(0)} D(x, ||x||),
\]
where \( D(y, r) \) is the disk centered at \( y \in \mathbb{R}^2 \) and of radius \( r \geq 0 \).

Theorem 1 provides an easy way to obtain the distribution of the area of \( \mathcal{F}(C(0)) \) conditionally to \( \{ N_0(C(0)) = k \} \), \( k \geq 3 \), and explicit integral formulas for the distribution of the area \( V_2(C) \) and the perimeter \( V_1(C) \) of \( C \).
Corollary 1 Conditionally to the event \( \{N_0(C) = k\} \), \( k \geq 3 \),
(i) the area \( V_2(\mathcal{F}(C(0))) \) is Gamma distributed of parameters \((k,1)\);
(ii) the distribution of \( V_2(C) \) is given by the following equality for every \( t \geq 0 \):
\[
P\{V_2(C) \geq t | N_0(C) = k\} = \int (1_{C_t} \cdot \varphi_k)(p_1, \ldots, p_k, \delta_1, \ldots, \delta_k) d\nu_k(p_1, \ldots, p_k, \delta_1, \ldots, \delta_k),
\]
where
\[
C_t = \{(p_1, \ldots, p_k, \delta_1, \ldots, \delta_k) \in (\mathbb{R}_+)^k \times (0, \pi)^k; \quad \frac{1}{8} \sum_{i=1}^{k} \frac{1}{\sin(\delta_{i-1}) \sin(\delta_i)} p_i (p_{i-1} \sin(\delta_i) + p_{i+1} \sin(\delta_{i-1}) - p_i \sin(\delta_{i-1} + \delta_i)) \geq t\};
\]
(iii) the distribution of \( V_1(C) \) is given by the following equality for every \( t \geq 0 \):
\[
P\{V_1(C) \geq t | N_0(C) = k\} = \int (1_{E_t} \cdot \varphi_k)(p_1, \ldots, p_k, \delta_1, \ldots, \delta_k) d\nu_k(p_1, \ldots, p_k, \delta_1, \ldots, \delta_k),
\]
where
\[
E_t = \{(p_1, \ldots, p_k, \delta_1, \ldots, \delta_k) \in (\mathbb{R}_+)^k \times (0, \pi)^k; \quad \frac{1}{2} \sum_{i=1}^{k} \frac{1}{\sin(\delta_{i-1}) \sin(\delta_i)} (p_{i-1} \sin(\delta_i) + p_{i+1} \sin(\delta_{i-1}) - p_i \sin(\delta_{i-1} + \delta_i)) \geq t\}.
\]

Remark 1 The point (i) was already obtained by Zuyev [23] with a different method based on Russo’s formula. The result can be easily extended to a \( d \)-dimensional Poisson-Voronoi tessellation, \( d \geq 3 \), in the following way: conditionally to the event \{number of hyperfaces of \( C(0) = k \)\}, \( k \geq d+1 \), the Lebesgue measure of the fundamental domain of \( C(0) \) is Gamma distributed of parameters \((k,1)\).

1.2 The Crofton cell of a Poisson line process.

Let us now consider \( \Phi' \) a Poisson point process in \( \mathbb{R}^2 \) of intensity measure
\[
\mu(A) = \int_{0}^{+\infty} \int_{0}^{2\pi} 1_A(r, u) d\theta dr, \quad A \in \mathcal{B}(\mathbb{R}^2).
\]
Let us consider for all \( x \in \mathbb{R}^2 \), \( H(x) = \{ y \in \mathbb{R}^2; (y-x) \cdot x = 0 \} \), \((x \cdot y)\) being the usual scalar product.
Then the set \( \mathcal{H} = \{ H(x); x \in \Phi \} \) is called a Poisson line process and divides the plane into convex polygons that constitute the so-called two-dimensional Poissonian tessellation. This tessellation is isotropic, i.e. invariant in law by any isometric transformation of the Euclidean space.
This random object was used for the first time by S. A. Goudsmit [8] and by R. E. Miles ([14], [15] and [16]). In particular, it provides a model for the fibrous structure of sheets of paper.
The origin is almost surely included in a unique cell \( C_0' \), called the Crofton cell. As in Theorem 1, we can get the joint distribution of the number of sides \( N_0(C_0') \) of \( C_0' \) and the respective positions of its bounding lines.
Theorem 2 (i) For every $k \geq 3$, we have

$$
P\{N_0(C'_0) = k\} = \frac{(2\pi)^k}{k!} \int d\sigma_k(\delta_1, \ldots, \delta_k) \int \prod_{i=1}^{k} e^{-p_i \left(\frac{1-\cos(\delta_i)}{\sin(\delta_i)} + \frac{1-\cos(\delta_{i-1})}{\sin(\delta_{i-1})}\right)} \mathbf{1}_{B}(p_{i-1}, p_i, p_{i+1}, \delta_{i-1}, \delta_i)dp_i; \quad (7)$$

(ii) conditionally to $\{N_0(C'_0) = k\}$, let us denote by $(P'_1, \Theta'_1), \ldots, (P'_k, \Theta'_k)$ the polar coordinates of the projections of the origin on the consecutive lines bounding $C'_0$ in the trigonometric order.

The joint distribution of the vector

$$(P'_1, \ldots, P'_k, \Theta'_2 - \Theta'_1, \ldots, \Theta'_{k-1} - \Theta'_k, 2\pi + \Theta'_1 - \Theta'_k)$$

then has a density with respect to the measure $\nu_k$ (defined by (5)) and its density $\varphi'_k$ is given by the following equality for every $p_1, \ldots, p_k \geq 0$, $(\delta_1, \ldots, \delta_k) \in \mathcal{S}_k$,

$$
\varphi'_k(p_1, \ldots, p_k, \delta_1, \ldots, \delta_k) = \frac{1}{P\{N_0(C'_0) = k\}} \frac{(2\pi)^k}{k!} \int \prod_{i=1}^{k} e^{-p_i \left(\frac{1-\cos(\delta_i)}{\sin(\delta_i)} + \frac{1-\cos(\delta_{i-1})}{\sin(\delta_{i-1})}\right)} \mathbf{1}_{B}(p_{i-1}, p_i, p_{i+1}, \delta_{i-1}, \delta_i).
$$

As for the Voronoi case, the point (i) of Theorem 2 provides numerical values estimated by a Monte-Carlo procedure which are listed in Table 1.

We deduce from Theorem 2 the joint distributions of the couples $(N_0(C'_0), V_1(C'_0))$ and $(N_0(C'_0), V_2(C'_0))$.

Corollary 2 Conditionally to the event $\{N_0(C'_0) = k\}$, $k \geq 3$,

(i) the perimeter $V_1(C'_0)$ is Gamma distributed of parameters $(k,1)$;

(ii) the distribution of $V_2(C'_0)$ is given by the following equality for every $t \geq 0$:

$$
P\{V_2(C'_0) \geq t | N_0(C'_0) = k\} = \int (1_{C_{t/4}} \cdot \varphi'_k)(p_1, \ldots, p_k, \delta_1, \ldots, \delta_k) d\nu_k(p_1, \ldots, p_k, \delta_1, \ldots, \delta_k),
$$

where the set $C_{t/4}$ is defined by the equality (6).

Remark 2 The point (i) was already obtained by G. Matheron (see [10], p.177). It can be extended to any $d$-dimensional Poissonian tessellation, $d \geq 3$, in the following way: conditionally to the event $\{\text{number of hyperfaces of } C'_0 = k\}$, $k \geq d + 1$, the mean width of $C'_0$ is Gamma distributed of parameters $\left(k, \frac{\Gamma(d/2)}{\pi^{d/2}}\right)$.

1.3 The typical cell of a Poisson line process.

The notion of typical (or empirical) cell $C'$ for the Poisson tessellation was first introduced by Miles [14], [15] through the convergence of ergodic means and has been reinterpreted since by means of a Palm measure (see [11], [12] and [4]). The typical cell $C'$ is connected in law to the Crofton cell by the following equality (see for example [4]):

$$
\mathbf{E}h(C') = \frac{1}{\mathbf{E}(1/V_2(C'_0))} \mathbf{E} \left( \frac{h(C'_0)}{V_2(C'_0)} \right), \quad (8)
$$
Theorem 3

(i) For every \( k \geq 3 \), we have

\[
\mathbb{P}\{N_0(C') = k\} = \frac{(2\pi)^k}{\pi \cdot k!} \int d\sigma_k(\delta_1, \cdots, \delta_k)
\]

\[
\int \prod_{i=1}^k e^{-p_i \left( \frac{1 - \cos(\delta_i)}{\sin(\delta_i)} + \frac{1 - \cos(\delta_{i-1})}{\sin(\delta_{i-1})} \right)} 1_B(p_{i-1}, p_i, p_{i+1}, \delta_{i-1}, \delta_i) W_k(p_1, \cdots, p_k, \delta_1, \cdots, \delta_k) dp_1 \cdots dp_k
\]

where

\[
W_k(p_1, \cdots, p_k, \delta_1, \cdots, \delta_k) = \frac{1}{2} \sum_{i=1}^k \frac{1}{\sin(\delta_{i-1}) \sin(\delta_i)} p_i (p_{i-1} \sin(\delta_i) + p_{i+1} \sin(\delta_{i-1}) - p_i \sin(\delta_{i-1} + \delta_i));
\]

(ii) Let

\[
(Q_1, \cdots, Q_k, \Sigma_1, \cdots, \Sigma_k) \in (\mathbb{R}_+)^k \times \mathcal{S}_k
\]

be a random vector which has a density \( \psi_k \) with respect to the measure \( \nu_k \) (given by (5)) satisfying the following equality for every \( p_1, \cdots, p_k \geq 0, (\delta_1, \cdots, \delta_k) \in \mathcal{S}_k,

\[
\psi_k(p_1, \cdots, p_k, \delta_1, \cdots, \delta_k) = a_k \cdot \frac{\varphi_k'(p_1, \cdots, p_k, \delta_1, \cdots, \delta_k)}{W_k(p_1, \cdots, p_k, \delta_1, \cdots, \delta_k)}.
\]

where \( a_k = (\mathbb{P}\{N_0(C') = k\})/(\pi \mathbb{P}\{N_0(C') = k\}) \).

Let us consider a random angle \( \Theta \) independent of the preceding vector and uniformly distributed on the circle. We denote by \( X_1, X_2, \cdots, X_k \) the points of the plane of respective polar coordinates \( (Q_1, \Theta), (Q_2, \Theta + \Sigma_1), \cdots, (Q_k, \Theta + \Sigma_1 + \cdots + \Sigma_{k-1}) \). The typical cell \( C' \) then is equal in law to the convex polygon bounded by the lines \( H(X_1), \cdots, H(X_k) \).

Numerical values for the distribution function of \( N_0(C') \) using the point (i) and a Monte-Carlo method are listed in Table 2. Let us remark that Miles [14] obtained that \( \mathbb{P}\{N_0(C') = 3\} = 2 - \frac{\pi^2}{6} \) and Tanner [21] get the exact value for \( \mathbb{P}\{N_0(C') = 4\} \).

As for the Crofton cell, we deduce from the preceding theorem a corollary about the joint distributions of the number of sides and the perimeter \( V_1(C') \) (resp. the area \( V_2(C') \)) of the typical cell.
<table>
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<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
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<tr>
<td>$\mathbf{P}{N_0(C') = k}$</td>
<td>0.3554</td>
<td>0.3815</td>
<td>0.1873</td>
<td>0.0596</td>
<td>0.0129</td>
<td>0.0023</td>
<td>0.0004</td>
</tr>
</tbody>
</table>

Table 2: Numerical values for $\mathbf{P}\{N_0(C') = k\}$.

**Corollary 3** Conditionally to the event $\{N_0(C') = k\}$, $k \geq 3$,

(i) the perimeter $V_1(C')$ is Gamma distributed of parameters $(k - 2, 1)$;

(ii) the distribution of $V_2(C')$ is given by the following equality for every $k \geq 3$, $t \geq 0$:

$$
\mathbf{P}\{V_2(C') \geq t \mid N_0(C') = k\} = \int (1_{C_{t/4} \cdot \psi_k})(p_1, \cdots, p_k, \delta_1, \cdots, \delta_k) d\nu_k(p_1, \cdots, p_k, \delta_1, \cdots, \delta_k),
$$

where the set $C_{t/4}$ is defined by the equality (6).

**Remark 3** The point (i) was already obtained by R. E. Miles [14]. It can be extended to any $d$-dimensional Poissonian tessellation in the following way: conditionally to the event $\{\text{number of hyperfaces of } C' = k\}$, $k \geq d + 1$, the mean width of $C'$ is Gamma distributed of parameters $(k - d, \frac{\Gamma(d/2)}{\pi^{d/2}})$.

**Remark 4** Comparing the points (i) of Corollaries 1 and 3, we notice that the area of the fundamental domain of $C(0)$ plays the same role for the Poisson-Voronoi case as the perimeter of $C_0$ for the Poisson line process. This analogy may be explained as follows: for every fixed measure in $\mathbb{R}^2$, the set of the lines $H(x)$, $x \in \mathbb{R}^2$, induces a pseudo-metric in the plane in the sense of R. V. Ambartzumian [1], [2]. The quantity $V_2(\mathcal{F}(C(0)))$ (resp. $V_1(C')$) then is proportional to the perimeter of the typical cell with respect to the pseudo-metric associated to the intensity measure of the Poisson point process $\Phi$ (resp. $\Phi'$).

In the paper, we first prove the results relative to the Poisson-Voronoi tessellation and secondly the analogous facts for the Crofton cell of a Poisson line process. Let us remark that Theorem 3 and Corollary 3 are direct consequences of Theorem 2 and Corollary 2 combined with (8) and (9).

### 2 Proofs of Theorem 1 and Corollary 1.

We use the same technique as in [5] based on Slivnyak’s formula (see e.g. [17]).

For every $x \in \mathbb{R}^2$, let us denote by $L(x)$ (respectively $\mathcal{D}(x)$) the bisecting line of the segment $[0, x]$ (respectively the half-plane containing 0 delimited by $L(x)$).

We then define for all $k \geq 3$, and $x_1, \cdots, x_k \in \mathbb{R}^2$, the domain

$$
\mathcal{D}(x_1, \cdots, x_k) = \cap_{i=1}^{k} \mathcal{D}(x_k).
$$

Besides, we consider the set of $(\mathbb{R}^2)^k$

$$
A_k = \{(x_1, \cdots, x_k) \in (\mathbb{R}^2)^k; \mathcal{D}(x_1, \cdots, x_k) \text{ is a convex polygon with } k \text{ sides}\},
$$

and for every $(x_1, \cdots, x_k) \in A_k$, the Lebesgue measure of the fundamental domain of $\mathcal{D}(x_1, \cdots, x_k)$, i.e.

$$
V(x_1, \cdots, x_k) = V_2(\mathcal{F}(\mathcal{D}(x_1, \cdots, x_k))) = V_2\left[\bigcup_{x \in \mathcal{D}(x_1, \cdots, x_k)} \mathcal{D}(x, ||x||)\right].
$$

Let $N_0$ be the set of all neighbors of the origin.
**Proposition 1** For every \( k \geq 3 \) and every bounded and measurable function \( h : \mathbb{R}^k \rightarrow \mathbb{R} \) invariant by permutation, we have

\[
E \left\{ 1_{\{N_0(C(0))=k\}} h(N_0) \right\} = \frac{1}{k!} \int h(x_1, \ldots, x_k) \exp\{-V(x_1, \ldots, x_k)\} 1_{A_k}(x_1, \ldots, x_k) dx_1 \cdots dx_k. \tag{12}
\]

**Proof.** Let us decompose \( \Omega \) over all possibilities for the set \( N_0 \).

\[
E \left\{ 1_{\{N_0(C(0))=k\}} h(N_0) \right\} = E \left\{ \sum_{\{x_1,\ldots,x_k\} \subset \Phi} h(x_1, \ldots, x_k) 1_{A_k}(x_1, \ldots, x_k) 1_{\{D(x_1,\ldots,x_k)=C(0)\}} \right\}
\]

Using Slivnyak’s formula [17], we obtain

\[
E \left\{ 1_{\{N_0(C(0))=k\}} h(N_0) \right\} = \frac{1}{k!} \int h(x_1, \ldots, x_k) 1_{A_k}(x_1, \ldots, x_k) E \left( 1_{\{L(y) \cap D(x_1,\ldots,x_k)=\emptyset \ \forall y \in \Phi\}} \right) dx_1 \cdots dx_k
\]

\[
= \frac{1}{k!} \int h(x_1, \ldots, x_k) 1_{A_k}(x_1, \ldots, x_k) P\{L(y) \cap D(x_1,\ldots,x_k)=\emptyset \ \forall y \in \Phi\} dx_1 \cdots dx_k. \tag{13}
\]

We can easily verify that for any \( z \in \mathbb{R}^2 \),

\[
L(z) \cap D(x_1, \ldots, x_k) \neq \emptyset \iff z \in \bigcup_{x \in D(x_1,\ldots,x_k)} D(x, ||x||),
\]

From this remark and the Poissonian property of \( \Phi \), we get

\[
P\{L(y) \cap D(x_1,\ldots,x_k)=\emptyset \ \forall y \in \Phi\} = P(\Phi \cap \bigcup_{x \in D(x_1,\ldots,x_k)} D(x, ||x||) = \emptyset)
\]

\[
= e^{-V(x_1,\ldots,x_k)}. \tag{14}
\]

Inserting the equality (14) in (13), we deduce Proposition 1.

\[\square\]

We already expressed the set \( A_k \) analytically and calculated the area \( V(x_1, \ldots, x_k) \) in function of the polar coordinates of \( x_1, \ldots, x_k \) (see [5], lemmas 1 and 2). Let us denote by

\[
(p_1, \theta_1), \ldots, (p_k, \theta_k) \in \mathbb{R}_+ \times [0, 2\pi),
\]

the respective polar coordinates of \( x_1, \ldots, x_k \in \mathbb{R}^2 \). Supposing that \( \theta_1, \ldots, \theta_k \) are in growing order, we define \( \delta_i = \theta_{i+1} - \theta_i, 1 \leq i \leq (k-1) \), and \( \delta_k = 2\pi + \theta_1 - \theta_k \). We then have the two following results:

\[
1_{A_k}(x_1, \ldots, x_k) = \prod_{i=1}^{k} 1_{B}(p_{i-1}, p_i, p_{i+1}, \delta_{i-1}, \delta_i), \tag{15}
\]
where the set $B$ is defined by (3), and for every $(x_1, \ldots, x_k) \in A_k$,

$$V(x_1, \ldots, x_k) = \sum_{i=1}^{k} \frac{1}{2 \sin^2(\delta_i)} \left\{ \left( p_i^2 + p_{i+1}^2 - 2p_i p_{i+1} \cos(\delta_i) \right) \frac{\delta_i}{2} + p_i p_{i+1} \sin(\delta_i) \right\} - \frac{p_i^2}{4} \sin(2\delta_i) - \frac{p_{i+1}^2}{4} \sin(2\delta_i) \right\}.$$  \hspace{1cm} (16)

**Proof of Theorem 1.** Using polar coordinates in the integral of the equality (12), we obtain for every $k \geq 3$,

$$E\{1_{\{N_0(c)=k\}} h(N_0)\} = \frac{1}{k!} \int e^{-V(p_1 u_1, \ldots, p_k u_k)} (h \cdot 1_{A_k})(p_1 u_1, \ldots, p_k u_k) \prod_{i=1}^{k} 1_{\{p_i \geq 0\}} 1_{\{0 \leq \theta_i \leq 2\pi\}} p_i dp_i d\theta_i$$

$$= \int e^{-V(p_1 u_1, \ldots, p_k u_k)} (h \cdot 1_{A_k})(p_1 u_1, \ldots, p_k u_k) 1_{\{0 \leq \theta_1 \leq \cdots \leq \theta_k \leq 2\pi\}} \prod_{i=1}^{k} 1_{\{p_i \geq 0\}} p_i dp_i d\theta_i, \hspace{1cm} (17)$$

where $u_\theta$, $0 \leq \theta \leq 2\pi$, denotes the unit vector in the plane of rectangular coordinates $(\cos \theta, \sin \theta)$. Let us suppose that $h$ is invariant under rotation, i.e. for all $\theta \in [0, 2\pi]$,

$$h(p_1 u_{\theta+\theta_1}, \cdots, p_k u_{\theta+\theta_k}) = h(p_1 u_1, \cdots, p_k u_k).$$

Inserting then the results (15) and (16) in (17), we deduce that

$$E\{1_{\{N_0(c)=k\}} h(N_0)\} = \left[ \int h(p_1 u_0, p_2 u_{\delta_1}, \cdots, p_k u_{\delta_1+\cdots+\delta_{k-1}}) \prod_{i=1}^{k} e^{-H(\delta_i, p_i, p_{i+1})} 1_{B(p_{i-1}, p_i, p_{i+1}, \delta_{i-1}, \delta_i)} p_i dp_i \right] \prod_{i=1}^{k} 1_{\{\delta_1+\cdots+\delta_{k-1} \leq 2\pi\}} d\delta_1 \cdots d\delta_{k-1}$$

$$= \frac{(2\pi)^k}{k!} \int d\sigma_k(\delta_1, \cdots, \delta_k) \int h(p_1 u_0, p_2 u_{\delta_1}, \cdots, p_k u_{\delta_1+\cdots+\delta_{k-1}}) \prod_{i=1}^{k} e^{-H(\delta_i, p_i, p_{i+1})} 1_{B(p_{i-1}, p_i, p_{i+1}, \delta_{i-1}, \delta_i)} p_i dp_i, \hspace{1cm} (18)$$

where the function $H$ is defined by the equality (4).

This last equality provides us the point (ii) of Theorem 1 and replacing $h$ by 1, we obtain the point (i).

\hspace{1cm} $\square$

**Proof of Corollary 1.** Let us first notice that for every $(x_1, \cdots, x_k) \in A_k$,

$$V_2(D(x_1, \cdots, x_k)) = \frac{1}{8} \sum_{i=1}^{k} \frac{1}{\sin(\delta_{i-1}) \sin(\delta_i)} p_i (p_{i-1} \sin(\delta_i) + p_{i+1} \sin(\delta_{i-1}) - p_i \sin(\delta_{i-1} + \delta_i)), \hspace{1cm} (19)$$

and

$$V_1(D(x_1, \cdots, x_k)) = \frac{1}{2} \sum_{i=1}^{k} \frac{1}{\sin(\delta_{i-1}) \sin(\delta_i)} (p_{i-1} \sin(\delta_i) + p_{i+1} \sin(\delta_{i-1}) - p_i \sin(\delta_{i-1} + \delta_i))^2 \hspace{1cm} (20)$$

$$= \frac{1}{2} \sum_{i=1}^{k} p_i \left( \frac{1 - \cos(\delta_i)}{\sin(\delta_i)} + \frac{1 - \cos(\delta_{i-1})}{\sin(\delta_{i-1})} \right). \hspace{1cm} (21)$$

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The point (ii) (resp. (iii)) then is easily obtained by applying the equality (18) to
\[ h(x_1, \ldots, x_k) = 1_{\{V_2(D(x_1, \ldots, x_k)) \geq t\}} \]
(resp. \( h(x_1, \ldots, x_k) = 1_{\{V_1(D(x_1, \ldots, x_k)) \geq t\}} \)). As for point (i), let us apply the equality (12) to
\[ h(x_1, \ldots, x_k) = e^{-\lambda V(x_1, \ldots, x_k)}, \quad \lambda \geq 0. \]

Let us notice that if \( N_0 = \{x_1, \ldots, x_k\} \), we have \( V(x_1, \ldots, x_k) = V_2(F(C(0))) \).

Consequently, we obtain
\[ \mathbb{E} \left\{ 1_{\{N_0(C(0)) = k\}} e^{-\lambda V_2(F(C(0)))} \right\} = \frac{1}{k!} \int e^{-(\lambda+1)V(x_1, \ldots, x_k)} 1_{A_k}(x_1, \ldots, x_k) dx_1 \cdots dx_k. \]

We take the change of variables \( x'_i = \sqrt{\lambda + 1} x_i, \ 1 \leq i \leq k, \) to deduce that
\[ \mathbb{E} \left\{ 1_{\{N_0(C(0)) = k\}} e^{-\lambda V_2(F(C(0)))} \right\} = \frac{1}{(\lambda + 1)^{k+1}} \frac{1}{k!} \int e^{-V(x_1, \ldots, x_k)} 1_{A_k}(x_1, \ldots, x_k) dx_1 \cdots dx_k = \mathbb{P}\{N_0(C(0)) = k\} \frac{1}{(\lambda + 1)^{k}}. \]

So conditionally to the event \( \{N_0(C(0)) = k\} \), the Laplace transform of the distribution of \( V_2(F(C(0))) \) is exactly \((\lambda + 1)^{-k}, \lambda \geq 0, \) i.e. \( V_2(F(C(0))) \) is Gamma distributed with parameters \((k, 1)\).

\[ \square \]

### 3 Proofs of Theorem 2 and Corollary 2.

For all \( x \in \mathbb{R}^2 \), let us define \( D'(x) \) as the half-plane containing the origin delimited by the line \( H(x) \). We then denote for every \( x_1, \ldots, x_k \in \mathbb{R}^2 \),
\[ D'(x_1, \ldots, x_k) = D'(x_1) \cap \cdots \cap D'(x_k) = D(2x_1, \ldots, 2x_k). \]

Let \( N'_0 \) be the (random) set of all points \( x \in \Phi' \) such that \( H(x) \) intersects the boundary of the Crofton cell \( C'_0 \).

**Proposition 2** For every \( k \geq 3 \) and every bounded and measurable function \( h : \mathbb{R}^k \rightarrow \mathbb{R} \) invariant by permutation, we have
\[ \mathbb{E} \left\{ 1_{\{N_0(C'_0) = k\}} h(N'_0) \right\} = \frac{1}{k!} \int (h \cdot 1_{A_k})(x_1, \ldots, x_k) \exp\{-V_1(D'(x_1, \ldots, x_k))\} dx_1 \cdots dx_k. \tag{22} \]

**Proof.** As for Proposition 1, we apply Slivnyak's formula to obtain
\[ \mathbb{E} \left\{ 1_{\{N_0(C'_0) = k\}} h(N'_0) \right\} = \frac{1}{k!} \int h(x_1, \ldots, x_k) 1_{A_k}(x_1, \ldots, x_k) \mathbb{P}\{H(y) \cap D'(x_1, \ldots, x_k) = \emptyset \ \forall y \in \Phi'\} dx_1 \cdots dx_k. \tag{23} \]

We can easily verify (see e.g. [7]) that
\[ \mathbb{P}\{H(y) \cap D'(x_1, \ldots, x_k) = \emptyset \ \forall y \in \Phi'\} = \mathbb{P}\{D'(x_1, \ldots, x_k) \subseteq C'_0\} = \exp\{-V_1(D'(x_1, \ldots, x_k))\}. \tag{24} \]

Inserting the equality (24) in (23), we deduce Proposition 2.
Proofs of Theorem 2 and Corollary 2. Let us recall that

$$V_1(D'(x_1, \cdots, x_k)) = \sum_{i=1}^{k} p_i \left( \frac{1 - \cos(\delta_i)}{\sin(\delta_i)} + \frac{1 - \cos(\delta_{i-1})}{\sin(\delta_{i-1})} \right), \quad (25)$$

and

$$V_2(D'(x_1, \cdots, x_k)) = \frac{1}{2} \sum_{i=1}^{k} \frac{1}{\sin(\delta_{i-1}) \sin(\delta_i)} p_i (p_{i-1} \sin(\delta_i) + p_{i+1} \sin(\delta_{i-1}) - p_i \sin(\delta_{i-1} + \delta_i)). \quad (26)$$

It then suffices to insert in (22) the results (15) and (25) to obtain the two points of Theorem 2.

The proof of Corollary 2 is also analogous to the Voronoi case. In particular, point (i) is deduced from a calculation of the Laplace transform of the distribution of the perimeter of $C'_0$ conditioned by the event $\{N_0(C'_0) = k\}, k \geq 3$:

$$\mathbb{E} \left\{ 1_{\{N_0(C'_0) = k\}} e^{-\lambda V_1(C'_0)} \right\} = \mathbb{P} \{N_0(C'_0) = k\} \cdot \frac{1}{(\lambda + 1)^k}, \quad \lambda \geq 0.$$

References


