Endpoint localization in Log Gamma polymer with boundary conditions

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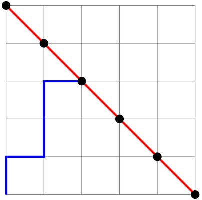
(Based on joint works with Francis Comets)

Polymer Model



A random walk in a random environment:

- A polymer path $x = (x_t; t = 0, \dots, n)$ is a nearest neighbour up-right path in $\mathbb{Z}_+^2$ of length n.
- The random environment $\omega = \{\omega(i,j) : (i,j) \in \mathbb{Z}_+^2\}$ sequence of real i.i.d random variables with probability distribution $\mathbb{P}$.

The point-to-line polymer measure of a path of length n is

$$Q_n^\omega(x) = \frac{1}{Z_n^\omega} \exp\left\{\beta \sum_{t=1}^{n} \omega(x_t)\right\}.$$

with Z_n^ω the sum over all up-right paths x starting at $x_0 = (0,0)$.
Question : For typical environment ω, what is the behaviour of the polymer of large size $n \to \infty$?

- $\beta = 0$, we obtain a simple random walk.
- If $\beta \to \infty$, Q_n concentrates on the path(s) that maximize $\sum \omega(x_k)$ and we obtain the last passage percolation model.

KPZ universality class

Recent developments on understanding models in KPZ universality class (Kardar, Parisi and Zhang 1986) which share the same scaling, statistics and limits objects which is related to the Tracy-Widom distributions.
KPZ class covers large random matrices, interacting particle systems (ASEP, TASEP,...), last passage percolation.
A few explicitly solvable polymer models:

- KPZ equation (Quastel ’10, Hairer ’13).
- Brownian semi-discrete polymer (O’Connell-Yor ’03).
- Log-gamma polymer (Seppalainen ’12), Strict-weak polymer (Corwin,Seppalainen,Shen ’14), Beta-polymer (Barraquand, Corwin’15).

Log gamma model

- We define the multiplicative weights as

$$Y_{i,j} = e^{\omega(i,j)},\ (i,j) \in {\mathbb{Z}_+}^2$$

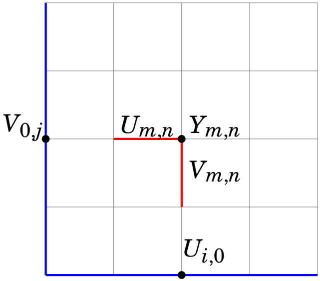
- We fix $\beta = 1$, the point-to-line partition function is

$$Z_n = \sum_{x_\cdot \in \Pi_n} \prod_{k=1}^{n} Y_{x_k},$$

Define Log-Gamma polymer with parameter $\mu > 0$, without boundary conditions

$$Y_{i,j}^{-1} \sim \Gamma(\mu)^{-1} x^{\mu-1} e^{-x} dx \qquad x > 0.$$

Log Gamma polymer with boundary conditions



Define : $U_{i,0} = Y_{i,0}$ and $V_{0,j} = Y_{0,j}$.
Model b.c.(θ): Let $\mu > 0$ be fixed. For $\theta \in (0, \mu)$, we will denote by b.c.(θ) the model with

- $U_{i,0}^{-1} \sim \text{Gamma}(\theta, 1)$
- $V_{0,j}^{-1} \sim \text{Gamma}(\mu - \theta, 1)$
- $Y_{i,j}^{-1} \sim \text{Gamma}(\mu, 1)$

Log-gamma polymer is in KPZ class

Seppalainen '12 discovered the stationarity property of this model with boundary conditions which makes it explicitly solvable:

- He obtains the value of the free energy

$$n^{-1}\ln Z_n \to -\Psi_0(\mu/2), \qquad \Psi_0 = \Gamma'/\Gamma$$

  and proves that the volume and wandering exponents for fluctuations are give by

$$\chi = 1/3, \qquad \xi = 2/3$$

- Later, explicit formula of the Laplace transform of the partition function at finite size is discovered by Corwin-O'Connell-Seppalainen-Zygouras '14 which implies GUE Tracy-Widom fluctuations for $\log(Z_{n,n})$ at scale $n^{1/3}$ (Borodin-Corwin-Remenik '13).

Localization in dimension 1

In order to analyse the localization phenomenon, we consider the largest probability for ending at a specific point:

$$I_n = \max_{x \in \mathbb{Z}^d} Q^{\omega}_{n-1}\{x_n = x\},$$

Carmona and Hu [2002] and Comets et al[2003] showed that, by using martingale methods, in dimension $d = 1$, there is a constant $c_0 = c_0(\beta) > 0$ such that the event

$$\limsup_{n \to \infty} I_n \geq c_0$$

has $\mathbb{P}$-probability one. This property is called endpoint localization.

What we will prove

The endpoint distribution under the quenched measure.

$$Q_n^\omega\{x_n = (k, n-k)\} = \frac{Z_{k,n-k}}{Z_n}, \quad k = 0, \ldots, n.$$

For each n, denote by

$$l_n = \arg\max\{Z_{k,n-k}, k \leq n\},$$

the location maximizing the above probability, and call it the "favourite endpoint". We will study the endpoint measure around the favourite point.

Proposition (Comets, Nguyen 2014)

Consider the model b.c.(θ) with $\theta \in (0,\mu)$. Define the end-point distribution $\tilde{\xi}^{(n)}$ centered around its mode, by

$$\tilde{\xi}^{(n)} = (\tilde{\xi}_k^{(n)}; k \in \mathbb{Z}), \quad \textit{with} \quad \tilde{\xi}_k^{(n)} = Q_n^{\omega}\{x_n = (l_n + k, n - l_n - k)\}.$$

Thus, $\tilde{\xi}^{(n)}$ is a random element of the set $\mathscr{M}_1$ of probability measures on $\mathbb{Z}$. Then, we have convergence in law

$$\tilde{\xi}^{(n)} \xrightarrow{\mathscr{L}} \xi \qquad \textit{in the space } (\mathscr{M}_1, \|\cdot\|_{TV}),$$

where $\|\mu - \nu\|_{TV} = \sum_k |\mu(k) - \nu(k)|$ is the total variation distance.

Remarks:

- The definition of ξ_k is given as a functional of the random walk conditioned to stay positive on $\mathbb{Z}^+$ and conditioned to stay strictly positive on $\mathbb{Z}^-$.
- The formula of ξ also depends on the case if $\theta = \mu/2$ (equilibrium case) or $\theta \neq \mu/2$ (non-equilibrium). In this talk, I will consider the more interesting case $\theta = \mu/2$.

Consequences

A few consequences:

- Mass of favourite endpoint converges

$$I_n = \max_x Q^{\omega}_{n-1}\{x_n = x\} \xrightarrow{\mathscr{L}} \max\{(\xi(k)+\xi(k+1))/2;\ k \in \mathbb{Z}\} > 0.$$

- Tightness of the endpoint : Letting $\overrightarrow{l_n} = (l_n, n-l_n)$,

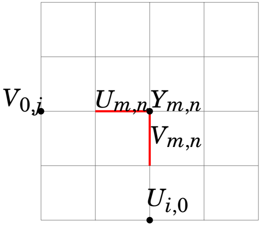
$$\lim_{K\to\infty} \limsup_{n\to\infty} Q^{\omega}_n \left[\|x_n - \overrightarrow{l_n}\| \geq K \right] = 0$$

- Scaling limit of endpoint: By Donsker's invariance principle, when $\theta = \mu/2$ we have

$$\frac{l_n}{n} \xrightarrow{\mathscr{L}} \arg \min_{t\in[0,1]W_t},$$

the arcsine law.

Stationary structure

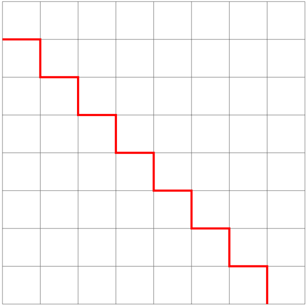


Recall the point-to-point partition function $Z_{m,n}$

$$Z_{m,n} = \sum_{x} \exp\Big\{\beta \sum_{t=1}^{m+n} \omega(x_t)\Big\}.$$

Compute $Z_{m,n}$ for all $(m,n) \in \mathbb{Z}_+^2$ and then define

$$U_{m,n} = \frac{Z_{m,n}}{Z_{m-1,n}}\ , \ V_{m,n} = \frac{Z_{m,n}}{Z_{m,n-1}}$$



The Burke's property of this model

Proposition (Seppalainen 2012)

Along any down-right path, the variables U, V's are mutually independent with marginal distributions

$$U^{-1} \sim Gamma(\theta, 1) \qquad V^{-1} \sim Gamma(\mu - \theta, 1)$$

Sketch of Proof.

Recall that $\theta = \mu/2$. Define for each $1 \le k \le n$, the random variable X_k^n

$$X_k^n = -\log(\frac{Z_{k,n-k}}{Z_{k-1,n-k+1}}) = -\log(\frac{U_{k,n-k}}{V_{k-1,n-k+1}}),$$

and $X_0^n = 0$. By the stationary structure, for each n, $(X_k^n)_{1\le k\le n}$ are i.i.d random variables with mean 0, and satisfy

$$\frac{Z_{k,n-k}}{Z_{0,n}} = \exp(-\sum_{i=0}^{k} X_i^n).$$

Defining $S_k^n = \sum_{i=1}^{k} X_i^n$ a random walk, then

$$Q_n^\omega\{x_n = (k,n-k)\} = \frac{Z_{k,n-k}}{\sum_{i=0}^{n} Z_{i,n-i}} = \frac{1}{\sum_{i=0}^{n} \exp(-(S_i^n - S_k^n))}$$

Since we are only interested in the law of $Q_n^\omega\{x_n = (k, n-k)\}$, we drop the superscript n in $S_n^n, X_i^n, \ldots$

$$S_n = \sum_{i=1}^{n} X_i,$$

and we define

$$\xi_k^n = \frac{1}{\sum_{i=0}^{n} \exp(-(S_i - S_k))}.$$

Then one can check that for every n:

$$(\xi_k^n)_{0 \le k \le n} \stackrel{L}{=} \left(Q_n^\omega\{x_n = (k, n-k)\}\right)_{0 \le k \le n},$$

Then we only need to prove that:

$$\{\xi^n_{\ell_n+k}\}_{k\in\mathbb{Z}} \xrightarrow{\mathscr{L}} \{\xi_k\}_{k\in\mathbb{Z}} \ , \qquad \text{in the } \ell_1-\text{norm},$$

with

$$\ell_n = \operatorname*{argmin}_{k\le n} S_k.$$

We show that the mass of the favourite point is converging :

$$\xi^n_{\ell_n} = \left(\sum_{i=0}^{n} e^{-(S_i - S_{\ell_n})}\right)^{-1} \xrightarrow{\mathscr{L}} \xi_0 \ , \qquad \text{in the } \ell_1-\text{norm},$$

The direct approach is to understand the growth of the random walk seen from its locals minima. This coupling is also the main tool to study the one dimension recurrent walk in random environment, discovered by Sinai and studied by Golosov.

Random walk conditioned to stay positive

Consider the random walk $S = (S_k, k \geq 0)$ in (16) has mean 0.
Define the event that the random walk stay non negative

$$\Lambda = \{S_k \geq 0 \text{ for all } k \geq 0\}.$$

we can approximate Λ with some other event Λ_n:

$$\Lambda_n = \{S_k \geq 0, \forall\ 0 \leq k \leq n\}.$$

And we would like to understand $\lim_{n\to\infty} \mathbb{E}(f(S)|\Lambda_n)$.

Proposition (Bertoin 1994)

For a bounded function $f(S) = f(S_1, \ldots, S_k)$*,*

$$\lim_{n \to \infty} \mathbb{E}(f(S) | \Lambda_n) = \mathbb{E}(f(S^{\uparrow})).$$

where $S^{\uparrow}$ is a homogeneous Markov chain on the nonnegative real numbers with transition function:

$$p^V(x,y) = \frac{V(y)}{V(x)} p(x,y) 1_{\{y \geq 0\}}.$$

and $V(x)$ is a renewal function

$$V(x) = 1 + \mathbb{E}\Big(\sum_{i=1}^{\sigma(0)-1} 1_{\{-x \leq S_i\}}\Big),$$

where

$$\sigma(0) = \min\{k \geq 1 : S_k \geq 0\}.$$

The process $S^{\uparrow}$ is known as the random walk conditioned to be positive.

Similarly for the reflected random walk $-S$, we can define the process $S^{\downarrow}$ and obtain a direct consequence:

Corollary

For fixed K, one has the following convergence results when $n \to \infty$:

$$(S_{\ell_n+k} - S_{\ell_n})_{1\le k\le K} \xrightarrow{\mathscr{L}} (S_k^{\uparrow})_{1\le k\le K} \ ,$$

$$(S_{\ell_n+k} - S_{\ell_n})_{-1\ge k\ge -K} \xrightarrow{\mathscr{L}} (S_k^{\downarrow})_{1\le k\le K} \ ,$$

$$\left(\sum_{k=-K}^{K} e^{-(S_{k+\ell_n} - S_{\ell_n})}\right)^{-1} \xrightarrow{\mathscr{L}} \left(1 + \sum_{k=1}^{K} e^{-S_k^{\uparrow}} + \sum_{k=1}^{K} e^{-S_k^{\downarrow}}\right)^{-1} \ .$$

At this point, we can get the explicit formula for ξ_0:

Lemma

For $n \to \infty$,

$$\xi_{\ell_n}^n = \left(\sum_{i=0}^{n} e^{-(S_i - S_{\ell_n})} \right)^{-1} \xrightarrow{\mathcal{L}} \xi_0 = \left(1 + \sum_{i=1}^{\infty} e^{-S_i^{\uparrow}} + \sum_{i=1}^{\infty} e^{-S_i^{\downarrow}} \right)^{-1} .$$

Now to control the full (unbounded) sum, we use a consequence of a result by Ritter '81:

$$\lim_{\delta \to 0} \mathbb{P}[S_{k+l_n} - S_{l_n} > \delta k^{1/2-\epsilon} \text{ for all } \; k \leq n - l_n] = 1$$

and we get the result.

For general value of k, we have :

$$\xi_k = \begin{cases} \dfrac{\exp(-S_k^{\uparrow})}{1+\sum_{i=1}^{\infty}\exp(-S_i^{\uparrow})+\sum_{i=1}^{\infty}\exp(-S_i^{\downarrow})}, & \text{if } k \geq 0 \\ \dfrac{\exp(-S_k^{\downarrow})}{1+\sum_{i=1}^{\infty}\exp(-S_i^{\uparrow})+\sum_{i=1}^{\infty}\exp(-S_i^{\downarrow})}, & \text{if } k < 0 \end{cases}$$

The case without boundary conditions

In this case, our method does not work and we need other approach.

- Macdonald process
- Combinatoric using Geometric RSK correspondence, Whittaker functions

For multipoint problem, we find formally the Airy process as limit [Nguyen, Zygouras'15]

Claim

Let $Z_{(m_1,n_1)}, Z_{(m_2,n_2)}$ *be point-to-point partition functions of a* $(0,\gamma)$*–log-gamma polymer with* $(m_1,n_1),(m_2,n_2)$ *determined by*

$$(m_1,n_1) = (N - t_1 N^{2/3}, N + t_1 N^{2/3}) \quad \textit{and} \quad (m_2,n_2) = (N + t_2 N^{2/3}, N - t_2 N^{2/3}),$$

Then

$$\left(\frac{\log Z_{(m_1,n_1)} - N f_\gamma}{(c_1^\gamma)^{-1} N^{1/3}}, \frac{\log Z_{(m_2,n_2)} - N f_\gamma}{(c_1^\gamma)^{-1} N^{1/3}} \right) \xrightarrow[N\to\infty]{(d)} \left(\mathrm{Ai}(-c_3^\gamma t_1) - c_2^\gamma t_1^2, \mathrm{Ai}(c_3^\gamma t_2) - c_2^\gamma t_2^2\right).$$