Hydrodynamic limits for the velocity-flip model

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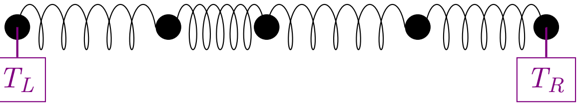
General idea

1. **Microscopic level:** System of N particles, interacting and evolving in time ($N \sim 10^{23}$).



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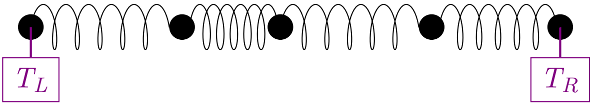

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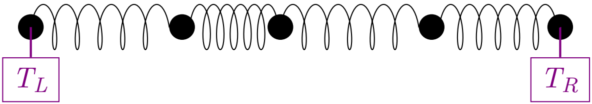

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- Equations (PDE's) on thermodynamical caracteristics: pressure, temperature $T(x,t)$,...
- Transport coefficients: $D(T)$ (diffusion).

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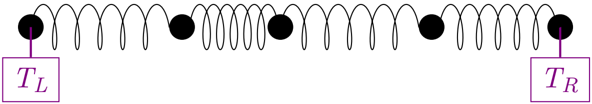

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$$\boxed{\frac{\partial T}{\partial t} = -\frac{\partial}{\partial x}\left[D(T)\frac{\partial T}{\partial x}\right]}$$

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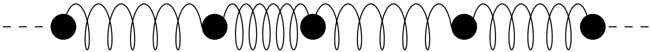
Main idea

Hydrodynamic limits = Propagation of local equilibrium

Model for heat conduction

- Chain of N harmonic coupled oscillators on the torus

$$\mathbb{T}_N = \{0, 1, ..., N-1\}$$



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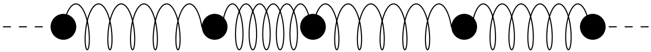

$$\begin{array}{lcl} p_x & : & \text{momentum of particle } x, \\ r_x & : & \text{distance between the particle } x \text{ and } x+1. \end{array}$$

- A typical configuration is $\omega = (\mathbf{r}, \mathbf{p}) \in (\mathbb{R} \times \mathbb{R})^{\mathbb{T}_N}$

$$\mathbf{r} = (r_x)_{x \in \mathbb{T}_N}, \qquad \mathbf{p} = (p_x)_{x \in \mathbb{T}_N}.$$

p_x	:	momentum of particle x,
r_x	:	distance between the particle x and $x+1$.

- An **Hamiltonian system** described by

$$\mathcal{H} = \sum_{x \in \mathbb{T}_N} \frac{p_x^2 + r_x^2}{2} = \sum_{x \in \mathbb{T}_N} e_x.$$

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- **Conserved quantities:**

$$\sum_{x \in \mathbb{T}_N} r_x \qquad \sum_{x \in \mathbb{T}_N} p_x \qquad \sum_{x \in \mathbb{T}_N} e_x$$

Evolution of the configuration $\omega(t)$

Configurations: $\omega_x(t) := \big(p_x(t), r_x(t)\big)$ in $\Omega_N := (\mathbb{R} \times \mathbb{R})^{\mathbb{T}_N}$.

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- **Generator:** for all $f : \Omega_N \to \mathbb{R}$, define $\mathcal{A}_N(f) : \Omega_N \to \mathbb{R}$ by

$$\mathcal{A}_N(f)(\omega) = F(\omega) \cdot \frac{\partial f}{\partial \omega}$$

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Explicitely,

$$\boxed{\mathcal{A}_N(f) = \sum_{x \in \mathbb{T}_N} (p_{x+1} - p_x) \cdot \frac{\partial f}{\partial r_x} + (r_x - r_{x-1}) \cdot \frac{\partial f}{\partial p_x}}$$

- We add a **stochastic noise** $\Rightarrow$ provides ergodicity.

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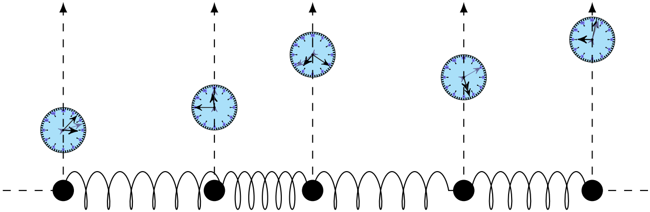
Each particle x waits independently a random Poissonian time and then flips p_x into $-p_x$.

The new configuration is denoted by $(\mathbf{r}, \mathbf{p}^x)$.



- The **generator** of the dynamics is:

$$\mathcal{L}_N = \mathcal{A}_N + \gamma S_N,$$

where

$$\boxed{\begin{aligned} \mathcal{A}_N(f) = & \quad \sum_{x \in \mathbb{T}_N} (p_{x+1} - p_x) \cdot \frac{\partial f}{\partial r_x} + (r_x - r_{x-1}) \cdot \frac{\partial f}{\partial p_x}, \\ \mathcal{S}_N(f)(\mathbf{r}, \mathbf{p}) = & \quad \frac{1}{2} \sum_{x \in \mathbb{T}_N} [f(\mathbf{r}, \mathbf{p}^x) - f(\mathbf{r}, \mathbf{p})]. \end{aligned}}$$

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- Family of **invariant measures**: the Gibbs states

$$\mu^N_{\beta,\lambda}(d\mathbf{r}, d\mathbf{p}) = \prod_{x\in\mathbb{T}_N} \frac{e^{-\beta e_x - \lambda r_x}}{Z(\beta,\lambda)} dr_x dp_x.$$

  - ▷ **Chemical potentials:** $\beta > 0$, $\lambda \in \mathbb{R}$,
  - ▷ **Hilbert space:** $\mathbb{L}^2(\mu^N_{\beta,\lambda})$.

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Theorem 1.1 [Fritz, Funaki, Lebowitz, 1994]

If ν is a probability measure on $(\mathbb{R} \times \mathbb{R})^{\mathbb{Z}}$ which

1. has finite entropy density,
2. is translation invariant,
3. is **stationary for the infinite dynamics**,

then ν is a **convex combination of (infinite) Gibbs states**.

Local equilibrium

The **Gibbs local equilibrium measures** associated to the profiles $\mathbf{r}$ and $\mathbf{e}$ defined on $\mathbb{T} = [0,1]$ are

$$\mu^N_{\beta(\cdot),\lambda(\cdot)}(d\mathbf{r}, d\mathbf{p}) = \prod_{x \in \mathbb{T}_N} \frac{\exp\left(-\beta\left(\frac{x}{N}\right) e_x - \lambda\left(\frac{x}{N}\right) r_x\right)}{Z_{\beta(\cdot),\lambda(\cdot)}} dr_x \, dp_x,$$

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where $\beta(\cdot)$ and $\lambda(\cdot)$ are the two chemical potential profiles related to $\mathbf{r}(\cdot)$ and $\mathbf{e}(\cdot)$ by the thermodynamical relations:

$$\begin{cases} \mathbf{e} = \dfrac{1}{\beta} + \dfrac{\lambda^2}{2\beta^2}, \\ \mathbf{r} = -\dfrac{\lambda}{\beta}. \end{cases}$$

Local equilibrium

- Initially,
  - ▷ a **local equilibrium** μ_0^N on $(\mathbb{R} \times \mathbb{R})^{\mathbb{T}_N}$, which is associated to a deformation profile $\mathbf{r}_0$ and an energy profile $\mathbf{e}_0$ on the torus $\mathbb{T} = [0, 1]$:

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For any continuous G on $\mathbb{T}$, $\delta > 0$,

$$\lim_{N\to\infty} \mu_0^N \left[\left| \frac{1}{N} \sum_{x\in\mathbb{T}_N} G\left(\frac{x}{N}\right) r_x - \int_{\mathbb{T}} G(v)\mathbf{r}_0(v)\,dv \right| > \delta \right] = 0.$$

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Diffusive scaling

- We study the process in the **diffusive scale**, at time tN^2:
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- The **relative entropy** of the probability law μ with respect to the probability law ν is

$$H(\mu|\nu) = \sup_f \left\{ \int_X f \; d\mu - \log\left(\int_X e^f \; d\nu \right) \right\}.$$

(the supremum is carried over all bounded functions f)

Theorem 2.1.

We suppose that μ_0^N is a Gibbs local equilibrium associated to the profiles $\mathbf{e}_0$ and $\mathbf{r}_0$:

$$\mu_0^N = \mu^N_{\beta_0(\cdot),\lambda_0(\cdot)}.$$

Then, μ_t^N is *close* to the Gibbs local equilibrium associated to the profiles $\mathbf{e}(t,\cdot)$ and $\mathbf{r}(t,\cdot)$ defined on $\mathbb{R}_+ \times \mathbb{T}$ and solutions of

$$\begin{cases} \dfrac{\partial \mathbf{r}}{\partial t} = \dfrac{1}{\gamma} \cdot \dfrac{\partial^2 \mathbf{r}}{\partial q^2}, \\ \dfrac{\partial \mathbf{e}}{\partial t} = \dfrac{1}{2\gamma} \cdot \dfrac{\partial^2}{\partial q^2} \left(\mathbf{e} + \dfrac{\mathbf{r}^2}{2} \right), \end{cases} \qquad \begin{cases} \mathbf{r}(0,q) = \mathbf{r}_0(q), \\ \mathbf{e}(0,q) = \mathbf{e}_0(q), \end{cases}$$

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in the sense:

$$H\left(\mu_t^N | \mu^N_{\beta(t,\cdot),\lambda(t,\cdot)}\right) = o(N)$$

Remark

- Conclusion $H_N(t) := H\left(\mu_t^N | \mu^N_{\beta(t,\cdot),\lambda(t,\cdot)}\right) = o(N)$ implies:

Hydrodynamic limit

For any continuous G on $\mathbb{T}$, "any" local function φ, and $\delta > 0$,

$$\mu_t^N\left[\left|\frac{1}{N}\sum_x G\left(\frac{x}{N}\right)\tau_x\varphi - \int_{\mathbb{T}} G(y)\ \tilde{\varphi}(\mathbf{e}(t,y),\mathbf{r}(t,y))\,dy\right| > \delta\right] \xrightarrow[N\to\infty]{} 0,$$

where $\tilde{\varphi}$ is the grand-canonical expectation of φ (under $\mu_{\beta,\lambda}$).

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- Key ingredient is **entropy inequality**: for all $\alpha > 0$,

$$\int g\ d\mu_t^N \leqslant \frac{H_N(t)}{\alpha} + \frac{1}{\alpha} \log\left(\int e^{\alpha g}\ d\mu^N_{\beta(t,\cdot),\lambda(t,\cdot)} \right)$$

What are the main issues?

1. **Non-gradient** system:

$$\frac{de_x}{dt} = [p_{x+2}r_{x+1} - p_{x+1}r_x](t) \ \text{ but } \ p_{x+1}r_x(t) \ \text{ is not a gradient.}$$

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2. **Second order** approximations for the relative entropy:

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3. **Large energies**:

$\Rightarrow$ *need to control all energy moments.*

How to prove the theorem?

- We correct the local Gibbs equilibrium:

$$d\nu_t^N = \frac{1}{Z_t} \prod_{x \in \mathbb{T}_N} e^{-\beta_t\left(\frac{x}{N}\right) e_x - \lambda_t\left(\frac{x}{N}\right) r_x + \frac{1}{N} F\left(t, \frac{x}{N}\right) \cdot \tau_x h} dr_x dp_x.$$

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[Funaki, Uchiyama, Yau, 1996; Olla, Tremoulet, 2002]

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- **Goal:** a Gronwall estimate for $H_N(t) := H\left(\mu_t^N | \nu_t^N\right)$:

$$\frac{dH_N(t)}{dt} \leqslant C \cdot H_N(t) + o(N).$$

  - ▷ All estimates are uniform in $t \in [0, T]$, T fixed.

Which techniques?

- Reduce the problem to densities: if $\phi_t^N = \dfrac{d\nu_t^N}{d\mu_{1,0}^N}$

$$\frac{dH_N(t)}{dt} \leqslant \int \left[\frac{1}{\phi_t^N} \left(N^2 \mathcal{L}_N^* \phi_t^N - \frac{d\phi_t^N}{dt} \right) \right] d\mu_t^N$$

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- Perform a Taylor expansion of

$$\frac{1}{\phi_t^N} \left(N^2 \mathcal{L}_N^* \phi_t^N - \frac{d\phi_t^N}{dt} \right)$$

with respect to $1/N$.

Fluctuation-dissipation equations

- Obtain the **fluctuation-dissipation equations**:

$$\mathcal{L}_N(r_x) = \nabla(p_{x+1}) \quad \text{and} \quad p_{x+1} = \nabla\left(f_x\right) + \mathcal{L}_N^*(g_x),$$

$$\mathcal{L}_N(e_x) = \nabla(p_{x+1} r_x) \quad \text{and} \quad p_{x+1} r_x = \nabla\left(h_x\right) + \mathcal{L}_N^*(k_x)$$

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One-block estimate

- Replace the average

$$\frac{1}{\ell}\sum_{x\in\Lambda_\ell(y)} p_x \qquad \text{by} \qquad \int p_0 \; d\mu^N_{\beta_\ell(y),\lambda_\ell(y)}$$

where $\beta_\ell(\cdot)$ and $\lambda_\ell(\cdot)$ are the profiles associated to the microscopic average energy and deformation profiles

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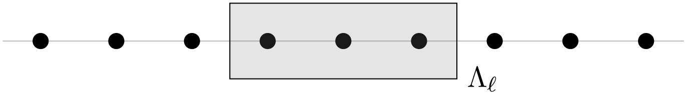

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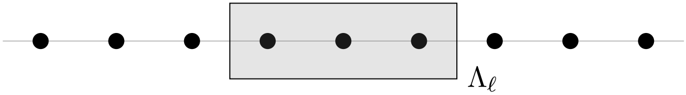

$\Rightarrow$ Standard technique from [**Olla, Varadhan, Yau (1993)**].

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- **What do we need?** Uniform control:

$$\forall\ k \geqslant 1,\ \int \left(\sum_{x \in \mathbb{T}_N} e_x^k \right) d\mu_t^N \leqslant C \cdot N,$$

where C is a constant which does not depend on N and t.

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Theorem 2.1

If μ_0^N is a local Gibbs equilibrium, then there exists a constant C that does not depend on N and t s.t.

$$\forall \; k \geqslant 1, \; \int \left(\sum_{x \in \mathbb{T}_N} e_x^k \right) d\mu_t^N \leqslant (C \cdot k)^k \cdot N.$$

Idea of the proof

- If μ_0^N is a local Gibbs equilibrium state $\mu^N_{\beta_0(\cdot),\lambda_0(\cdot)}$, then μ_t^N is a convex combination of Gaussian measures

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where ρ^t is the law of the r.v. $(m_t, C_t) \in \mathbb{R}^{2N} \times \mathfrak{S}_{2N}(\mathbb{R})$:

  - ▷ $(m_t, C_t)_{t \geqslant 0}$ is an explicit Markov process given by $(\omega_t)_{t\geqslant 0}$,
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[Bernardin, Kannan, Lebowitz, Lukkarinen, 2011]

- Consequently,

$$\int \left(\sum_{x \in \mathbb{T}_N} e_x^k \right) d\mu_t^N = \int G_{m,C} \left(\sum_{x \in \mathbb{T}_N} e_x^k \right) \; d\rho^t(m, C).$$

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- We are reduced to estimate

$$G_{m,C} \left(\sum_{x \in \mathbb{T}_N} e_x^k \right) := \int \left(\sum_{x \in \mathbb{T}_N} e_x^k \right) dG_{m,C},$$

"easily computable" thanks to the process (m_t, C_t).

Final theorem

Let μ_0^N be a **convex combination of Gibbs local equilibria**, *close* to the local Gibbs equilibrium associated to $\mathbf{e}_0$ and $\mathbf{r}_0$, in the sense:

$$H\left(\mu_0^N | \mu^N_{\beta_0(\cdot),\lambda_0(\cdot)}\right) = o(N)$$

Then, μ_t^N is *close* to the Gibbs local equilibrium associated to the profiles $\mathbf{e}(t,\cdot)$ and $\mathbf{r}(t,\cdot)$ defined on $\mathbb{R}_+ \times \mathbb{T}$ and solutions of

$$\begin{cases} \dfrac{\partial \mathbf{r}}{\partial t} = \dfrac{1}{\gamma} \cdot \dfrac{\partial^2 \mathbf{r}}{\partial q^2}, \\ \dfrac{\partial \mathbf{e}}{\partial t} = \dfrac{1}{2\gamma} \cdot \dfrac{\partial^2}{\partial q^2}\left(\mathbf{e} + \dfrac{\mathbf{r}^2}{2}\right), \end{cases} \qquad \begin{cases} \mathbf{r}(0,q) = \mathbf{r}_0(q), \\ \mathbf{e}(0,q) = \mathbf{e}_0(q), \end{cases}$$

in the sense:

$$H\left(\mu_t^N | \mu^N_{\beta(t,\cdot),\lambda(t,\cdot)}\right) = o(N)$$

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  - ▷ Add a random mass m_x on each atom x.

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- Macroscopic fluctuations?
  - ▷ e.g. large deviations for the current of energy.

References

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Thank you for your attention.