RANDOM ISING SYSTEMS

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- Brief history, euristic arguments
- The Imry-Ma argument (1975)
- The Bricmont-Kupianen result (1988)
- The Aizemnan-Wehr result (1990)
- Stability of the interfaces
- Kac ‘random field $d = 1$. Summary of previous results, COP (99), COPV (2005) and weak large deviations principle (OP, in preparation).
- Results on the dynamics associated to Kac ‘random field, $d \geq 3$, MOS (2003), MO (2005).

THE MODEL

- $\sigma \in \mathcal{S} = \{-1, +1\}^{Z^d}$ $(\mathcal{S}, \mathcal{F}, \rho)$, $\theta > 0$, $\Lambda \subset \mathbb{Z}^d$
  $\rho_i(\sigma_i = 1) = \rho_i(\sigma_i = -1) = \frac{1}{2}$
- Energy.

$$H_\Lambda(\sigma) = -\sum_{|i-j|=1, i \wedge j \in \Lambda} \sigma_i \sigma_j + \theta \sum_{i \in \Lambda} h_i(\omega) \sigma_i$$

$(\Omega, \mathcal{B}, \mathbb{P})$ (Polish space)

$$\mathbb{P}[h_i(\omega) = 1] = \mathbb{P}[h_i(\omega) = -1] = \frac{1}{2} \qquad i.i.r.v.$$

Given $\eta \in \mathcal{S}$, $\omega \in \Omega$, the random finite volume Gibbs measure:

$$\mu^{\eta,\omega}_{\beta,\Lambda}(d\sigma_\Lambda) = \frac{1}{Z^{\eta,\omega}_\Lambda} e^{-\beta H_\Lambda(\sigma_\Lambda,\eta_{\Lambda^c})} \rho_\Lambda(d\sigma_\Lambda)\delta_{\eta_{\Lambda^c}}(d\sigma_{\Lambda^c})$$

As function of ω is a measurable function. Care should be taken when performing the limit.

Heuristic argument

- The cumulative effect of the magnetic field on a uniform spin configuration in a region $[-L, L]^d$ is $\sim L^{\frac{d}{2}}$
- The symmetry breaking mechanism is

$$|\partial\Lambda| = L^{d-1}$$

- The critical dimension is $d = 2$
- Bricmont, Kupianen (1988) $d \geq 3$: existence of phase transition for temperature and disorder small
- Aizenman, Wehr (1990) $d = 2$ a.s. unicity of the Gibbs measure.

The Imry-Ma argument (1975)

- They tried to extend the Peierls argument to a situation where the symmetry was broken by the presence of the random field.
- Peierls argument : Basic intuition: for large β (low temperature), the Gibbs measures should strongly favor configurations with minimal energy.
- If $h \neq 0$, then the configuration $\sigma_i = sign(h)$ will be the configuration with minimal energy. If $h = 0$ then $\sigma_i \equiv 1$ or $\sigma_i \equiv -1$ will be the two minimal states.
- introduce contours, which characterize locally unlikely configurations

- show that typical configurations do not contain large regions where configurations are atypical
- $$\mu(0 \in \dot{\Gamma}) \leq e^{-2\beta|\Gamma|} \qquad |\Gamma| \quad \text{finite}$$

  essential the spin-flip symmetry
- there are several choices for configurations not containing large undesiderable regions

$$|\{\Gamma : |\Gamma| = k, 0 \in \dot{\Gamma}\}| \leq 3^k$$

$$\mu_\beta(\exists \Gamma \in \Gamma(\sigma) : 0 \in \dot{\Gamma}) \leq \sum_{k \geq 2d} e^{-2\beta k} 3^k < \frac{1}{2}$$

if β large enough. This implies the existence of at least two different Gibbs measures.

- In the RFIM the bulk energies of the two ground states are not the same. If the $\sigma_i = 1$, $i \in \dot{\Gamma}$

$$E_{\text{bulk}}(\Gamma) = -\theta \sum_{i \in \dot{\Gamma}} h_i$$

If the $\sigma_i = -1$, $i \in \dot{\Gamma}$

$$E_{\text{bulk}}(\Gamma) = \theta \sum_{i \in \dot{\Gamma}} h_i$$

- $$\mu(0 \in \dot{\Gamma}) \simeq e^{-2\beta|\Gamma|} e^{2\theta \sum_{i \in \dot{\Gamma}} h_i}$$

When h_i are bounded

$$|2\theta \sum_{i \in \dot{\Gamma}} h_i| \leq 2\theta |\dot{\Gamma}|$$

If $|\dot{\Gamma}|$ is large, even if θ small, can be bigger than surface term.
Imry-Ma argue that the "typical value" for

$$E_{\text{bulk}}(\Gamma) \simeq \pm\theta \sqrt{|\dot{\Gamma}|} \simeq \pm\theta |\Gamma|^{\frac{d}{2(d-1)}},$$

since

$$|\dot{\Gamma}| \leq 2d|\Gamma|^{\frac{d}{(d-1)}}$$

- Then

$$\mu(0 \in \dot{\Gamma}) \simeq e^{-2|\Gamma|\left[\beta - \theta|\Gamma|^{\frac{2-d}{2(d-1)}}\right]}.$$

  Small when $d > 2$ and θ small.
- These considerations led Imry-Ma to the **correct** prediction.
- **Remark** If θ is not small, then even in small contours (no CLT) the bulk energy can dominate surface energy.
- Following Imry-Ma we repeat their argument in a “more” precise way.

- Apply Peierls argument. Γ a contour containing the origin. Denote Γ^{int} the inner boundary and Γ^{out} the exterior boundary

$$\mu^{+1}_{\dot{\Gamma},\beta}[\sigma_{\Gamma^{int}} = -1] \leq e^{-2\beta|\Gamma|} \frac{Z^{-1}_{\dot{\Gamma}\backslash\Gamma^{int},\beta}}{Z^{+1}_{\dot{\Gamma}\backslash\Gamma^{int},\beta}}$$

$$\mu^{-1}_{\dot{\Gamma},\beta}[\sigma_{\Gamma^{int}} = +1] \leq e^{-2\beta|\Gamma|} \frac{Z^{+1}_{\dot{\Gamma}\backslash\Gamma^{int},\beta}}{Z^{-1}_{\dot{\Gamma}\backslash\Gamma^{int},\beta}}$$

Lemma *In the RFIM, for any Gibbs state* μ_β*,*

$$\mu_\beta[\Gamma \in \Gamma(\sigma)] \leq e^{\{-2\beta|\Gamma| + |\ln Z^{+1}_{\dot{\Gamma}\backslash\Gamma^{int},\beta} - Z^{-1}_{\dot{\Gamma}\backslash\Gamma^{int},\beta}|\}}$$

- To treat the last term we use the **concentration of measures** phenomenon.
- Roughly: a Lipschitz function of i.i.d. random variables has fluctuations not bigger than those of a corresponding linear function. Investigated widely in the last 30 years (see Ledoux -Talagrand).

- **Theorem** *Let* $f : [-1,1]^N \to \mathbb{R}$ *be a function whose level sets are convex. Suppose that* f *Lipschitz,* $X, Y \in [-1,1]^N$*,*

$$|f(X) - f(Y)| \leq C_{Lip} \|X - Y\|_2$$

Then if $X_1, ..X_N$ *are i.i.d. random variables with values in* $[-1,1]$ *set* $Z = f(X_1, ..X_N)$

$$\mathbb{P}[|Z - M_Z| > z] \leq 4exp\left(-\frac{z^2}{16C_{Lip}^2}\right)$$

- **Comments**:

$$|\mathbb{E}(Z) - M_Z| \leq 8\sqrt{\pi}C_{Lip} \simeq C_{Lip}$$

$$\mathbb{P}[|Z - \mathbb{E}[Z]| > z] \leq 4exp\left(-\frac{(z - C_{Lip})^2}{16C_{Lip}^2}\right).$$

  Then, when C_{Lip} is small compared to z^2, one can replace M_z with $\mathbb{E}[Z]$.

- If X_i, $i = 1, \dots, N$ are i.i.d, standard gaussian random variable, then the estimate can be improved.

- **Lemma** *Assume the random field symmetric, bounded (or Gaussian distributed)*

$$\mathbb{P}[|\ln Z^{+1}_{\dot{\Gamma}\setminus\Gamma^{int}} - \ln Z^{-1}_{\dot{\Gamma}\setminus\Gamma^{int}}| \geq z] \leq Cexp\left(-\frac{z^2}{\theta^2\beta^2 C|\dot{\Gamma}|}\right)$$

- **proof** By symmetry of the distribution of h

$$\mathbb{E}[\ln Z^{+1}_{\dot{\Gamma}\setminus\Gamma^{int}}] = \mathbb{E}[\ln Z^{-1}_{\dot{\Gamma}\setminus\Gamma^{int}}]$$

$$\mathbb{P}[|\ln Z^{+1}_{\dot{\Gamma}\setminus\Gamma^{int}} - Z^{-1}_{\dot{\Gamma}\setminus\Gamma^{int}}| \geq z] \leq$$

$$\mathbb{P}[|\ln Z^{+1}_{\dot{\Gamma}\setminus\Gamma^{int}} - \mathbb{E}[\ln Z^{+1}_{\dot{\Gamma}\setminus\Gamma^{int}}]|$$

$$+ |\ln Z^{-1}_{\dot{\Gamma}\setminus\Gamma^{int}} - \mathbb{E}[\ln Z^{-1}_{\dot{\Gamma}\setminus\Gamma^{int}}]| \geq z]$$

- $$\leq 2\mathbb{P}[|\ln Z^{+1}_{\dot{\Gamma}\setminus\Gamma^{int}} - \mathbb{E}[\ln Z^{+1}_{\dot{\Gamma}\setminus\Gamma^{int}}]| \geq \frac{z}{2}]$$
- Let
$$f(h_1, \ldots, h_N) = |\ln Z^{+1}_{\dot{\Gamma}\setminus\Gamma^{int}}|$$
$$N \equiv |\dot{\Gamma} \setminus \Gamma^{int}|,$$
- f is convex (by differentiation)
- compute the Lipschitz norm

$$|f(h) - f(h')| \leq \sup_{\tilde{h}} \left| \sum_{j \in \dot{\Gamma} \setminus \Gamma^{int}} [h_j - h'_j] \frac{\partial [\ln Z^{+1}_{\dot{\Gamma} \setminus \Gamma^{int}, \beta}}{\partial h_j} (\tilde{h}) \right|$$

$$\leq \theta \beta \sup_{j \in \dot{\Gamma} \setminus \Gamma^{int}} \mu_{\dot{\Gamma} \setminus \Gamma^{int}, \beta} (\sigma_j) \left| \sum_{j \in \dot{\Gamma} \setminus \Gamma^{int}} [h_j - h'_j] \right|$$

$$\leq \theta \beta \sqrt{|\dot{\Gamma}|} \|h - h'\|_2 \quad \text{by Schwarz inequality.}$$

By previous Theorem we obtain the thesis.

- So for a given contour Γ, setting $z = \beta\theta\sqrt{|\dot{\Gamma}|}$

$$\mu_\beta[\Gamma \in \Gamma(\sigma)] \leq e^{\{-2\beta|\Gamma| + \theta\beta\sqrt{|\dot{\Gamma}|}\}}, \qquad \sqrt{|\dot{\Gamma}|} = |\Gamma|^{\frac{d}{2(d-1)}}$$

- with

$$\mathbb{P} \geq 1 - Ce^{-\frac{|\Gamma|^2}{|\dot{\Gamma}|C}} \geq 1 - C\exp\left\{-\frac{1}{C}|\Gamma|^{\frac{d-2}{d-1}}\right\}$$

since

$$\frac{|\Gamma|^2}{|\dot{\Gamma}|} = |\Gamma|^{\frac{d-2}{d-1}}$$

- One should prove that

$$\mathbb{P}\left[\mu_\beta[\exists\Gamma, \Gamma \in \Gamma(\sigma), 0 \in \dot{\Gamma}] < \frac{1}{2}\right] > 0$$

- This is the analogous of Peierls Theorem for random model. It implies that there will be at least two extremal Gibbs states with positive probability. However the number of extremal Gibbs states for a given random interactions with sufficient ergodic properties is almost sure constant (Newmann 97- Lectures in Mathematics).

- The attempt to show, for small θ,

$$\mathbb{P}\left[\exists\Gamma, 0 \in \dot{\Gamma}, |\ln Z^{+1}_{\dot{\Gamma}\setminus\Gamma^{int}} - \ln Z^{-1}_{\dot{\Gamma}\setminus\Gamma^{int}}| \geq \beta|\Gamma|\right]$$

is small **fails**

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$$\mathbb{P}\left[\exists\Gamma, 0 \in \dot{\Gamma}, |\ln Z^{+1}_{\dot{\Gamma}\setminus\Gamma^{int}} - \ln Z^{-1}_{\dot{\Gamma}\setminus\Gamma^{int}}| \geq \beta|\Gamma|\right]$$

$$\leq \sum_{\Gamma, 0\in\dot{\Gamma}} \mathbb{P}\left[|\ln Z^{+1}_{\dot{\Gamma}\setminus\Gamma^{int}} - \ln Z^{-1}_{\dot{\Gamma}\setminus\Gamma^{int}}| \geq \beta|\Gamma|\right]$$

$$\leq \sum_{\Gamma, 0 \in \dot{\Gamma}} exp\{-\frac{|\Gamma|^2}{C\theta^2|\dot{\Gamma}|}\} \qquad \text{since} \qquad \frac{|\Gamma|^2}{|\dot{\Gamma}|} = |\Gamma|^{\frac{d-2}{d-1}}$$

$$\leq \sum_{k \geq 2d} exp\{-\frac{1}{\theta^2}k^{\frac{d-2}{d-1}} + k \ln C\}$$

since

$$|\{\Gamma : |\Gamma| = k, 0 \in \dot{\Gamma}\}| \leq e^{k64\frac{\log d}{d}} \leq C^k$$

conclusion The sum diverges. The first inequality spoil the estimates!!!

- Reasonable if the partition function for different Γ were almost independent !! If Γ and Γ' are very similar this is not the case.
- **Theorem** *Assume that there exists $C > 0$ so that for all $\Lambda, \Lambda' \subset \mathbb{Z}^d$,*

$$\mathbb{P}\left[\left|\ln Z_{\Lambda}^{+1} - \ln Z_{\Lambda'}^{+1} - \mathbb{E}\left[\ln Z_{\Lambda}^{+1} - \ln Z_{\Lambda'}^{+1}\right]\right| \geq z\right]$$

$$\leq \exp\left(-\frac{z^2}{C\theta^2\beta^2|\Lambda\Delta\Lambda'|}\right)$$

then, if $d \geq 3$ there exists $\theta_0 > 0$, $\beta_0 < \infty$, so that for $\theta < \theta_0$ and $\beta \geq \beta_0$, $\mathbb{P}$ a.a. there exists at least two extremal infinite volume Gibbs state μ_β^+ and μ_b^-.

- The proof can be found in Bovier Note. It is based on *chaining* tecniques, used by Fisher-Frolich-Spencer (JSP 34 (1984)) in a model with “no contours within contours”.

$$\ln Z^{+1}_{\dot{\Gamma}\setminus\Gamma^{int}} - \ln Z^{-1}_{\dot{\Gamma}\setminus\Gamma^{int}} \simeq \sum_{i\in\dot{\Gamma}\setminus\Gamma^{int}} h_i$$

- Can we verify the hypothesis of previous Theorem?
- Suppose we compute the Lipschtiz norm

$$
\left|\ln Z_{\Lambda}^{+1}[h] - \ln Z_{\Lambda}^{+1}[h'] - \ln Z_{\Lambda'}^{+1}[h] + \ln Z_{\Lambda'}^{+1}[h']\right|
$$

$$
\leq \sup_{\tilde{h}} \left| \sum_{i \in \Lambda \setminus (\Lambda \cap \Lambda')} (h_i - h_i') \frac{\partial \ln Z_{\Lambda}^{+1}}{\partial h_i}[\tilde{h}] \right|
$$

$$
+ \left| \sum_{i \in \Lambda' \setminus (\Lambda \cap \Lambda')} (h_i - h_i') \frac{\partial \ln Z_{\Lambda'}^{+1}}{\partial h_i}[\tilde{h}] \right|
$$

$$
+ \left| \sum_{i \in \Lambda \cap \Lambda'} (h_i - h_i') \left(\frac{\partial \ln Z_{\Lambda}^{+1}}{\partial h_i}[\tilde{h}] - \frac{\partial \ln Z_{\Lambda'}^{+1}}{\partial h_i}[\tilde{h}] \right) \right|
$$

$$
\leq \theta \beta \sum_{i \in \Lambda \Delta \Lambda'} |h_i - h_i'| +
$$

$$+\theta\beta \sup_{\tilde{h}} \left| \sum_{i\in\Lambda\cap\Lambda'} (h_i - h_i') \left(\mu^+_\Lambda[\tilde{h}](\sigma_i) - \mu^+_{\Lambda'}[\tilde{h}](\sigma_i) \right) \right|$$

$$\leq \theta\beta \sqrt{|\Lambda\Delta\Lambda'|} \|h_{\Lambda\Delta\Lambda'} - h'_{\Lambda\Delta\Lambda'}\|_2 + \theta\beta \times$$

$$\sup_{\tilde{h}} \sqrt{\sum_{i\in\Lambda\cap\Lambda'} \left(\mu^+_\Lambda[\tilde{h}](\sigma_i) - \mu^+_{\Lambda'}[\tilde{h}](\sigma_i) \right)} \|h_{\Lambda\cap\Lambda'} - h'_{\Lambda\cap\Lambda'}\|_2$$

comment: the expectation of σ_i for $i \in \Lambda \cap \Lambda'$ should be almost the same, so one should prove that the last line is $\simeq |\Lambda\Delta\Lambda'|$.

- If one shows this, then this would be an alternative proof of phase transitions for IRFM in $d = 3$. The only rigorous proof of phase transitions in $d = 3$ is BK.
- Theorem [BK] *Let* $d \geq 3$*, assume* h_i *i.i.d. random variables,* $\mathbb{P}[|h_i| \geq h] \leq e^{-\frac{h^2}{s^2}}$ *for* s *sufficiently small. Then there exists* $\beta_0 < \infty$*,* $S_0 > 0$*, so that for* $\beta \geq \beta_0$*,* $s \leq S_0$*,* $\mu^{\pm}_{\Lambda_n,\beta}$ *converge to disjoint Gibbs measures* $\mu^{\pm}_{\beta}$*.*
- The proof is based on the RG analysis.

Absence of phase transitions: Aizenman-Wehr method

- The Imry-Ma argument predicts that the random bulk energies might overcome the surface terms. Then the particular realization of the random fields determines locally the orientation of the spins.
- The effects of the boundary conditions are not felt in the interior of the system.
- This implies an unique Gibbs state. Rigorously proven by A-W(1990).
- The proof is based on the RG analysis.

- Here the argument (roughly):
- Fix $\Lambda \subset \mathbb{Z}^2$. Consider the difference of the free energy

$$f_{\beta,\Lambda^+} - f_{\beta,\Lambda^-} = \ln \frac{Z^+_{\beta,\Lambda}}{Z^-_{\beta,\Lambda}} \leq C|\partial\Lambda|$$

$$f_{\beta,\Lambda^+} - f_{\beta,\Lambda^-} \geq C(\beta)\sqrt{|\Lambda|}W$$

W standard gaussian variable.

- Then

$$C(\beta)\sqrt{|\Lambda|}W \leq C|\partial\Lambda|$$

In $d = 2$, this implies $C(\beta) = 0$.

- $C(\beta)$ is linked to an order parameter: magnetization.
- $C(\beta) = 0$ implies $m = 0$, the uniqueness of the Gibbs state.
- I will specialize the proof only in the IRFM (the one described).

- Translation covariant states
- Let $(\Omega, \mathcal{B}, \mathbb{P})$ the probability space. Let T the translation group Z^d on Ω. We assume that $\mathbb{P}$ is invariant under the action of T.
  $(\Omega, \mathcal{B}, \mathbb{P}, T)$ is stationary and ergodic.
- In the case h_i i.i.d. random variables, stationarity and ergodicity are trivially satisfied.
- The action of T is

$$(h_{x_1}[T_y\omega], \ldots, h_{x_n}[T_y\omega]) \equiv (h_{x_1+y}[\omega], \ldots, h_{x_n+y}[\omega])$$

- We use that Ω has an affine structure:

$$(h_{x_1}[\omega + \omega'], \ldots, h_{x_n}[\omega + \omega'])$$
$$= (h_{x_1}[\omega] + h_{x_1}[\omega'], \ldots, h_{x_n}[\omega] + h_{x_n}[\omega'])$$

- Define

$$\Omega_0 \equiv \{\delta\omega \in \Omega : \exists \Lambda \subset \mathbb{Z}^d, \forall y \neq \Lambda, h_y[\delta\omega] = 0\}$$

Λ finite

- A random Gibbs measure μ_β is called covariant if
- $\forall x \in \mathbb{Z}^d$, f continuous,

$$\mu_\beta[\omega](T_{-x}f) = \mu_\beta[T_x\omega](f) \quad a.s.$$

- $\forall \delta\omega \in \Omega_0$, for a.a. ω

$$\mu_\beta[\omega + \delta\omega](f) = \frac{\mu_\beta[\omega]\left(f e^{-\beta[H(\omega+\delta\omega)-H(\omega)]}\right)}{\mu_\beta[\omega]\left(e^{-\beta[H(\omega+\delta\omega)-H(\omega)]}\right)}$$

Note that if $\delta\omega$ is supported on Λ

$$[H(\omega + \delta\omega) - H(\omega)] = -\sum_{i\in\Lambda} \sigma_i h_i[\delta\omega]$$

- **Theorem** *Consider the RFIM. Then there exists two covariant random Gibbs measures* $\mu_\beta^+[\omega]$ *and* $\mu_\beta^-[\omega]$ *so that*

$$\mu_\beta^\pm[\omega] = \lim_{\Lambda \uparrow \mathbb{Z}^d} \mu_{\Lambda,\beta}^\pm[\omega] \quad a.a.\omega$$

Suppose that for some β, $\mu_\beta^+[\omega] = \mu_\beta^-[\omega]$. *Then for this fixed value of* β *the Gibbs measure for RFIM is unique for a.a.* ω.

- **proof** By FKG (Fortuin, Kasteleyn, Ginibre) $\forall \omega \in \Omega$, one constructs $\mu_\beta^\pm[\omega]$ as limits of local specifications with constant boundary conditions along arbitrary (ω independent) increasing and absorbing sequences of finite volumes Λ_n.

- The function $\omega \to \mu_\beta^\pm[\omega]$ are measurable (limit of measurable functions).
- Are covariant states?
- Property (1) requires the independence of the limit from the chosen sequence Λ_n, which holds in the case at hand. We have

$$\mu^+_{\Lambda,\beta}[\omega](T_{-x}f) = \mu^+_{\Lambda+x,\beta}[T_x\omega](f)$$

  then $\mu_\beta^+[\omega](T_{-x}f) = \mu_\beta^+[T_x\omega](f)$. (consequence of FKG inequalities)
- Properties (2) holds trivially for local specifications with Λ large enough to support $\delta\omega$.

- Result for system satisfying FKG

$$\lim_{n\to\infty} \mu^{+}_{\beta,\Lambda_n} = \mu^{+}_{\beta} \qquad \Lambda_n \subset \mathbb{Z}^d$$

  exists and it is independent on the particular sequence. Same holds for μ^{-}_{β}.
- μ^{+}_{β} and μ^{-}_{β} are extremal Gibbs measures.
- $\mu^{-}_{\beta}(f) \leq \mu_{\beta}(f) \leq \mu^{+}_{\beta}(f)$
  where f is any increasing bounded continuous function.

- **Note** If FKG do not hold, then two difficulties
- measurability of the limit. This can be solved introducing the “metastates”
- one needs comparison results between local specifications in different volumes.

- Order parameters
  The monotonicity properties of FKG inequalities imply that **if** $\mu_\beta^+[\omega] = \mu_\beta^-[\omega]$ $\mathbb{P}$ a.s., **then** there exists an unique Gibbs state $\mathbb{P}$ a.s.
- Further in the translational invariant case, uniqueness implies that the total magnetization vanishes.
  Set the total magnetization

$$m^\mu[\omega] = \lim_{|\Lambda|\uparrow\infty} \frac{1}{|\Lambda|} \sum_{i\in\Lambda} \mu[\omega](\sigma_i)$$

- **Lemma** *Suppose* μ *a covariant Gibbs state. Then,* $\mathbb{P}$*a.s.* $m^\mu[\omega]$ *exists and it is independent on* ω.
- **proof:** By the covariance of μ,

$$m^\mu[\omega] = \lim_{|\Lambda|\to\infty} \frac{1}{|\Lambda|} \sum_{i\in\Lambda} \mu[T_{-i}\omega](\sigma_0)$$

- The $\mu[\omega](\sigma_0)$ is a bounded measurable function of ω.
- The since $(\Omega, \mathcal{B}, \mathbb{P}, T)$ is stationary and ergodic, the limit exists $\mathbb{P}$a.s. and

$$m^\mu = \mathbb{E}[\mu](\sigma_0)].$$

- **Lemma** *In the RFIM*

$$m^+ - m^- = 0 \iff \mu_\beta^+ = \mu_\beta^-$$

- proof: The $m^\mu = \mathbb{E}[\mu](\sigma_0)]$ implies that $\mathbb{P}$ a.s.

$$0 = m^+ - m^- = \mathbb{E}[\mu^+(\sigma_i) - \mu^-(\sigma_i)]$$

- Since $\mu^+(\sigma_i) - \mu^-(\sigma_i) \geq 0$ and there are only countable many sites i, almost surely for all $i \in \mathbb{Z}^d$

$$\mu^+(\sigma_i) = \mu^-(\sigma_i).$$

- Generating functions
- Set ($\theta = 1$)

$$G_\Lambda^\mu \equiv \frac{1}{\beta} \ln \mu[\omega] \left(e^{\beta \sum_{i\in\Lambda} h_i \sigma_i} \right).$$

- Note: if μ is a covariant state

$$G_\Lambda^{\mu[\omega]} \equiv -\frac{1}{\beta} \ln \mu[\omega - \omega_\Lambda] \left(e^{-\beta \sum_{i\in\Lambda} h_i \sigma_i} \right)$$

where ω_Λ is such that $h_i[\omega_\Lambda] = h_i[\omega]$ if $i \in \Lambda$, $h_i[\omega_\Lambda] = 0$ if $i \in \Lambda^c$

- $$\frac{\partial}{\partial h_i} G_\Lambda^{\mu[\omega]} = \frac{\mu[\omega - \omega_\Lambda](\sigma_i e^{-\beta \sum_{i \in \Lambda} h_i \sigma_i})}{\mu[\omega - \omega_\Lambda](e^{-\beta \sum_{i \in \Lambda} h_i \sigma_i})} = \mu[\omega](\sigma_i)$$

- Then

$$\mathbb{E}\left[\frac{\partial}{\partial h_i} G_\Lambda^{\mu^\pm[\cdot]}\right] = m^\pm.$$

- Define:

$$F_\Lambda = \mathbb{E}\left[G_\Lambda^{\mu^+} - G_\Lambda^{\mu^-} | \mathcal{B}_\Lambda\right].$$

$$\mathbb{E}\left[\frac{\partial}{\partial h_0} F_\Lambda\right] = m^+ - m^-.$$

- Important to show the following bound.
  Lemma *For all β and volume Λ*

$$|F_\Lambda| \leq 2|\partial\Lambda|.$$

- proof: First step: express F_Λ in terms of measures that do not depend on the disorder within Λ.

$$F_\Lambda = \frac{1}{\beta}\mathbb{E}\left[\ln \frac{\mu^+[\omega]\left(e^{\beta\sum_{i\in\Lambda} h_i\sigma_i}\right)}{\mu^-[\omega]\left(e^{\beta\sum_{i\in\Lambda} h_i\sigma_i}\right)}|\mathcal{B}_\Lambda\right]$$

$$= \mathbb{E}\left[\ln \frac{\mu^-[\omega-\omega_\Lambda]\left(e^{-\beta\sum_{i\in\Lambda} h_i\sigma_i}\right)}{\mu^+[\omega-\omega_\Lambda]\left(e^{-\beta\sum_{i\in\Lambda} h_i\sigma_i}\right)}|\mathcal{B}_\Lambda\right]$$

- Use spin-flip symmetry and symmetry of the distribution of h

$$\mu^+[\omega]\,(f(\sigma)) = \mu^-[-\omega]\,(f(-\sigma))\,*$$

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$$\mathbb{E}\left[\ln \frac{\mu^-[\omega-\omega_\Lambda]\left(e^{-\beta\sum_{i\in\Lambda}h_i\sigma_i}\right)}{\mu^+[\omega-\omega_\Lambda]\left(e^{-\beta\sum_{i\in\Lambda}h_i\sigma_i}\right)}|\mathcal{B}_\Lambda\right] \qquad \text{since } *$$

$$= \mathbb{E}\left[\ln \frac{\mu^+[-\omega+\omega_\Lambda]\left(e^{\beta\sum_{i\in\Lambda}h_i\sigma_i}\right)}{\mu^+[\omega-\omega_\Lambda]\left(e^{-\beta\sum_{i\in\Lambda}h_i\sigma_i}\right)}|\mathcal{B}_\Lambda\right]$$

$$= \mathbb{E}\left[\ln \frac{\mu^+[\omega-\omega_\Lambda]\left(e^{\beta\sum_{i\in\Lambda}h_i\sigma_i}\right)}{\mu^+[\omega-\omega_\Lambda]\left(e^{-\beta\sum_{i\in\Lambda}h_i\sigma_i}\right)}|\mathcal{B}_\Lambda\right]$$

- We have the ratio of two expectations with respect to the same measure.

$$\mu^+[\omega - \omega_\Lambda]\left(e^{\beta\sum_{i\in\Lambda} h_i\sigma_i}\right) =$$

$$\sum_{\sigma_{\Lambda^c}} \frac{\mu^+[\omega-\omega_\Lambda](\sigma_{\Lambda^c})}{Z_\Lambda^{\sigma_{\Lambda^c}}}$$

$$\times \sum_{\sigma_\Lambda} e^{\beta(\sum_{i,j\in\Lambda}\sigma_i\sigma_j + \sum_{i\in\Lambda, j\in\Lambda^c}\sigma_i\sigma_j + \sum_{i\in\Lambda} h_i\sigma_i)}$$

$$= \sum_{\sigma_{\Lambda^c}} \frac{\mu^+[\omega-\omega_\Lambda](\sigma_{\Lambda^c})}{Z_\Lambda^{\sigma_{\Lambda^c}}}$$

$$\times \sum_{\sigma_\Lambda} e^{\beta(\sum_{i,j\in\Lambda}\sigma_i\sigma_j - \sum_{i\in\Lambda, j\in\Lambda^c}\sigma_i\sigma_j - \sum_{i\in\Lambda} h_i\sigma_i)}$$

$$\leq e^{2\beta|\partial\Lambda|} \times$$

$$\sum_{\sigma_{\Lambda^c}} \frac{\mu^+[\omega - \omega_\Lambda](\sigma_{\Lambda^c})}{Z_\Lambda^{\sigma_{\Lambda^c}}}$$

$$\times \sum_{\sigma_\Lambda} e^{\beta(\sum_{i,j\in\Lambda} \sigma_i\sigma_j + \sum_{i\in\Lambda, j\in\Lambda^c} \sigma_i\sigma_j - \sum_{i\in\Lambda} h_i\sigma_i)}$$

$$= e^{2\beta|\partial\Lambda|} \mu^+[\omega - \omega_\Lambda] \left(e^{-\beta \sum_{i\in\Lambda} h_i\sigma_i} \right).$$

- **Lower bound on the Laplace transform of F_Λ**
Lemma *Suppose that for some $z > 0$, the distributions of h satisfies $\mathbb{E}[|h|^{2+z}] \leq C$. Then*

$$\liminf_{\Lambda, |\Lambda| \uparrow \infty} \mathbb{E}\left[exp\{\frac{tF_\Lambda}{\sqrt{\Lambda}}\}\right] \geq exp\{\frac{t^2 b^2}{2}\}$$

$$b^2 \geq \mathbb{E}\left[\mathbb{E}[F_\Lambda | \mathcal{B}_0]^2\right].$$

- The last two lemmas contradict each other in $d \leq 2$, *unless* $b = 0$.
- But $b = 0$ implies $m = 0$ then the uniqueness of the Gibbs state.

- **proof:** Order the points $x \in \Lambda$ in lexicographic order.

$$\mathcal{B}_{\Lambda,i}, \quad \sigma - \text{ algebra generated by} \{h_{x_j}\}_{x_j \in \Lambda, x_j \leq x_i}$$

-

$$F_\Lambda = \sum_{i=1}^{|\Lambda|} (\mathbb{E}[F_\Lambda | \mathcal{B}_{\Lambda,i}] - \mathbb{E}[F_\Lambda | \mathcal{B}_{\Lambda,i-1}]) \equiv \sum_{i=1}^{|\Lambda|} Y_i$$

- where

$$Y_i \equiv \mathbb{E}[F_\Lambda | \mathcal{B}_{\Lambda,i}] - \mathbb{E}[F_\Lambda | \mathcal{B}_{\Lambda,i-1}]$$

- Then

$$\mathbb{E}\left[exp\{tF_\Lambda\}\right] = \mathbb{E}\left[...\mathbb{E}\left[e^{tY_2}|\mathcal{B}_{\Lambda,1}\right] e^{tY_1}\right]$$

- We need a lower bound on terms

$$\mathbb{E}\left[e^{tY_i}|\mathcal{B}_{\Lambda,i-1}\right].$$

- We use the following: $\forall x \in \mathbb{R}$, $a \geq 0$ there exists $g(a) \downarrow 0$, when $a \uparrow 0$

$$e^x \geq 1 + x + \frac{1}{2}(1 - g(a))x^2 \mathbb{1}_{|x| \leq a}$$

$$1 + \frac{x^2}{2} \geq e^{\frac{1}{2}x^2 e^{-\frac{a^2}{2}}} \quad \text{when} \quad |x| \leq a.$$

- $\mathbb{E}[X] = 0$, set $f(a) = (1 - g(a))e^{-\frac{a^2}{2}}$ we obtain

$$\mathbb{E}[e^X] \geq e^{\frac{1}{2}f(a)\mathbb{E}[X^2 \mathbb{1}_{|X| \leq a}]}.$$

- We obtain

$$\mathbb{E}\left[e^{tY_i} | \mathcal{B}_{\Lambda,i-1}\right] e^{\left(-\frac{t^2}{2} f(a)\mathbb{E}\left[Y_i^2 \mathbb{1}_{\{t|Y_i| \leq a\}} || \mathcal{B}_{\Lambda,i-1}\right]\right)} \geq 1$$

- The last quantity is $\mathcal{B}_{\Lambda,i}$ measurable, then repeating

- $$\mathbb{E}\left[e^{\frac{tF_\Lambda}{\sqrt{|\Lambda|}}-\frac{t^2}{2}f(a)V_\Lambda(a)}\right]\geq 1$$

- where

$$V_\Lambda(a)\equiv\frac{1}{|\Lambda|}\sum_{i=1}^{|\Lambda|}\mathbb{E}\left[Y_i^2\mathbb{I}_{\{t|Y_i|\leq a\sqrt{|\Lambda|}\}}|\mathcal{B}_{\Lambda,i-1}\right].$$

- Suppose that there exists C **independent** on a:

$$\lim_{|\Lambda|\to\infty}V_\Lambda(a)=C \qquad \text{in probability.}$$

- Since in $d = 2$, $|F_\Lambda| \leq 2|\partial\Lambda| = 2\sqrt{|\Lambda|}$

$$\liminf_{\Lambda \uparrow \mathbb{Z}^2} \mathbb{E}\left[e^{\frac{tF_\Lambda}{\sqrt{|\Lambda|}}}\right] \geq e^{\frac{t^2}{2}C}$$

- Namely, for $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$

$$\mathbb{E}\left[e^{\frac{tF_\Lambda}{\sqrt{|\Lambda|}} - \frac{t^2}{2}f(a)V_\Lambda(a)}\right]$$

$$\leq \left(\mathbb{E}\left[e^{p\frac{tF_\Lambda}{\sqrt{|\Lambda|}}}\right]\right)^{\frac{1}{p}} \left(\mathbb{E}\left[e^{-q\frac{t^2}{2}f(a)V_\Lambda(a)}\right]\right)^{\frac{1}{q}}$$

$$\limsup_{\Lambda} \left(\mathbb{E}\left[e^{-q\frac{t^2}{2}f(a)V_\Lambda(a)} \right] \right)^{\frac{1}{q}} \leq e^{-\frac{t^2}{2}f(a)C}$$

Since $\lim_{a\to 0} f(a) = 1$,

$$\limsup_{a\to 0} \left(\limsup_{\Lambda} \left(\mathbb{E}\left[e^{-q\frac{t^2}{2}f(a)V_\Lambda(a)} \right] \right)^{\frac{1}{q}} \right) \leq e^{-\frac{t^2}{2}C}$$

Then

$$\liminf_{\Lambda\uparrow\mathbb{Z}^2} \mathbb{E}\left[e^{p\frac{tF_\Lambda}{\sqrt{|\Lambda|}}} \right] \geq e^{p\frac{t^2}{2}C}$$

Changing variable $t' = tp$

$$\liminf_{\Lambda \uparrow \mathbb{Z}^2} \mathbb{E}\left[e^{\frac{t' F_\Lambda}{\sqrt{|\Lambda|}}} \right] \geq e^{\frac{(t')^2}{2p} C} \qquad \forall p > 1$$

Then when $p \to 1$ the result!!

to show: $V_\Lambda(a)$ **admits limit and to identify it.**

We apply the ergodic Theorem.

$$\mathcal{B}_i^{\leq} = \sigma\text{-algebra generated by } h_j, j \leq i$$

$\leq$ refers to lexicographic ordering. Define

$$W_i \equiv \mathbb{E}\left[G_\Lambda^{\mu^+} - G_\Lambda^{\mu^-} | \mathcal{B}_i^{\leq}\right] - \mathbb{E}\left[G_\Lambda^{\mu^+} - G_\Lambda^{\mu^-} | \mathcal{B}_{i-1}^{\leq}\right]$$

We have

$$Y_i = \mathbb{E}[W_i | \mathcal{B}_\Lambda], \qquad i \in \Lambda$$

Important W_i is shift covariant, i.e.

$$W_i[\omega] = W_0[T_{-i}\omega].$$

By ergodic theorem one has

$$\lim_{|\Lambda|\uparrow\infty} \frac{1}{|\Lambda|} \sum_{i=1}^{|\Lambda|} \mathbb{E}\left[W_i^2 | \mathcal{B}_{i-1}^{\leq}\right] = \mathbb{E}\left[W_0^2\right] \qquad \text{in prob}$$

comment: Apply ergodic theorem to

$$f(\omega) = \mathbb{E}\left[W_0^2 | \mathcal{B}_0^{<}\right]$$

then

$$\mathbb{E}\left[W_i^2 | \mathcal{B}_{i-1}^{\leq}\right] = \mathbb{E}\left[W_i^2 | \mathcal{B}_i^{<}\right] = f(T_{-i}\omega)$$

Further we show that

$$\lim_{|\Lambda|\uparrow\infty} \mathbb{P}\left[\frac{1}{|\Lambda|}\sum_{i=1}^{|\Lambda|}\mathbb{E}\left[Y_i^2 \mathbb{I}_{\{t|Y_i|>a\sqrt{|\Lambda|}\}}|\mathcal{B}_{\Lambda,i-1}\right] > z\right] = 0$$

By Chebyshev inequality, compute

$$\mathbb{E}\left[\frac{1}{|\Lambda|}\sum_{i=1}^{|\Lambda|}\mathbb{E}\left[Y_i^2 \mathbb{I}_{\{t|Y_i|>a\sqrt{|\Lambda|}\}}|\mathcal{B}_{\Lambda,i-1}\right]\right]$$

$$= \frac{1}{|\Lambda|}\sum_{i=1}^{|\Lambda|}\mathbb{E}\left[Y_i^2 \mathbb{I}_{\{t|Y_i|>a\sqrt{|\Lambda|}\}}\right] \leq$$

$$\leq \frac{1}{|\Lambda|} \sum_{i=1}^{|\Lambda|} (\mathbb{E}\left[Y_i^{2q}\right])^{\frac{1}{q}} \left(\mathbb{P}\left[|Y_i| > \frac{a}{t}\sqrt{|\Lambda|}\right]\right)^{\frac{1}{p}}$$

where

$$\frac{1}{q} + \frac{1}{p} = 1.$$

We have that

$$\mathbb{E}\left[Y_i^{2q}\right] = \mathbb{E}\left[\mathbb{E}[W_i|\mathcal{B}_\Lambda]^{2q}\right] \leq \mathbb{E}\left[W_0^{2q}\right]$$

by Jensen inequality

One shows

$$|W_0| \leq C|h_0|$$

If the $2q$ moments of h are finite then $\mathbb{E}\left[W_0^{2q}\right] \leq C$.

By Chebyshev inequality

$$\mathbb{P}\left[|Y_i| > \frac{a}{t}\sqrt{|\Lambda|}\right] \leq \frac{t^2 \mathbb{E}[W_0^2]}{a^2 |\Lambda|^2}$$

Then

$$\mathbb{E}\left[\frac{1}{|\Lambda|}\sum_{i=1}^{|\Lambda|}\mathbb{E}\left[Y_i^2\mathbb{I}_{\{t|Y_i|>a\sqrt{|\Lambda|}\}}|\mathcal{B}_{\Lambda,i-1}\right]\right]$$

$$\leq \frac{t^2\mathbb{E}[W_0^2]}{a^2|\Lambda|^2}$$

which goes to 0 when $|\Lambda| \to \infty$. The proof is done if

$$\mathbb{E}\left[Y_i^2|\mathcal{B}_{\Lambda,i-1}\right] - \mathbb{E}\left[W_i^2|\mathcal{B}_{i-1}^{\leq}\right] \to 0 \qquad \text{in prob}$$

This can be shown easily:

$$\mathbb{E}\left\{\left(\mathbb{E}\left[Y_i^2|\mathcal{B}_{\Lambda,i-1}\right]-\mathbb{E}\left[W_i^2|\mathcal{B}_{i-1}^{\leq}\right]\right)^2\right\}\to 0$$

Namely

$$\mathbb{E}\left[Y_i^2|\mathcal{B}_{\Lambda,i-1}\right]=\mathbb{E}\left[\left(\mathbb{E}[W_i|\mathcal{B}_\Lambda\right)^2|\mathcal{B}_{\Lambda,i-1}\right]$$
$$\leq\mathbb{E}\left[\mathbb{E}[W_i^2|\mathcal{B}_\Lambda]|\mathcal{B}_{\Lambda,i-1}\right]=\mathbb{E}\left[\mathbb{E}[W_i^2|\mathcal{B}_{i-1}^{\leq}]|\mathcal{B}_\Lambda\right]$$

Denote $f=\mathbb{E}[W_i^2|\mathcal{B}_{i-1}^{\leq}]$.

Then since

$$\lim_{|\Lambda|\to\infty} \mathbb{E}\left[(f - \mathbb{E}[f|\mathcal{B}_\Lambda])^2\right] \to 0$$

we obtain the thesis.
Further, let $\mathcal{B}_0$ the sigma-algebra generated by the single variable h_0

$$\mathbb{E}[W_0^2] = \mathbb{E}[\mathbb{E}[W_0^2|\mathcal{B}_0]] \geq \mathbb{E}\left[(\mathbb{E}[W_0|\mathcal{B}_0])^2\right]$$

$$\mathbb{E}[W_0|\mathcal{B}_0] = \mathbb{E}[\mathbb{E}\left[G_\Lambda^{\mu^+} - G_\Lambda^{\mu^-}|\mathcal{B}_0^{\leq}\right]|\mathcal{B}_0] = \mathbb{E}[F_\Lambda|\mathcal{B}_0]$$

since

$$\mathbb{E}\left[\mathbb{E}\left[G_\Lambda^{\mu^+} - G_\Lambda^{\mu^-}|\mathcal{B}_{-1}^{\leq}\right]|\mathcal{B}_0\right] = \mathbb{E}\left[G_\Lambda^{\mu^+} - G_\Lambda^{\mu^-}\right] = 0$$

Finally observe that

$$\frac{\partial}{\partial h_0}\mathbb{E}[F_\Lambda|\mathcal{B}_0] = \mathbb{E}[\mu^+(\sigma_0) - \mu^-(\sigma_0)|\mathcal{B}_0]$$

Set

$$f(h) = \mathbb{E}[F_\Lambda|\mathcal{B}_0] \qquad \text{where} \qquad h = h_0[\omega].$$

If $\mathbb{E}\left[f^2\right] = 0$, then

$$f(h) = 0 \qquad \mathbb{P}a.s.$$

on the support of the distribution of h. Since

$$0 \leq f'(h) \leq 1$$

$$f'(h) = 0 \qquad \mathbb{P}a.s$$

then

$$\mu^+ - \mu^- = 0.$$

Theorem [AW] *In the IRFM, whose distribution is not concetrated on a single point and possesses at least* $2 + z$ *finite moments, for some* $z > 0$*, if* $d \leq 2$ *there exists an unique infinite-volume Gibbs state.*

Comments Soft result. It does not say anything more precise about the properties of the Gibbs state. How does the Gibbs state at high temperature distinguish itself from the one at low temperature?

At low temperature for large θ the Gibbs state will concentrate near configurations $\sigma_i = sign h_i$. For small θ more complicated behaviour is expected.

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