Noise induced escape problem and phase reduction

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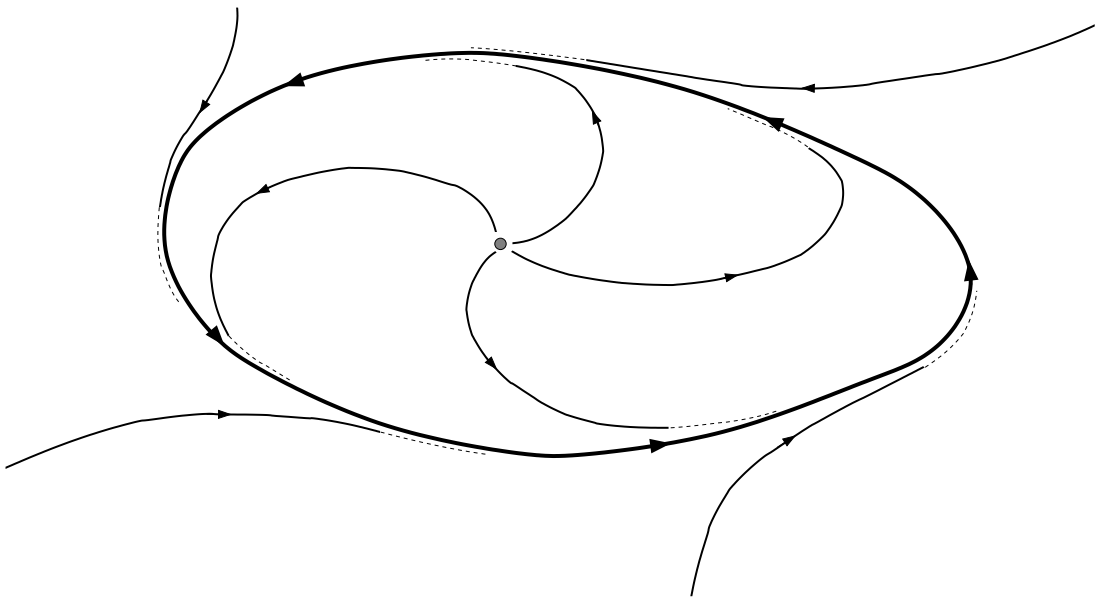
Simple example of phase reduction

Consider a dynamical system in $\mathbb{R}^n$ of the type

$$\dot{X}_t = F(X_t)$$

admitting a limit cycle.



On the limit cycle, it reduces to a phase dynamics

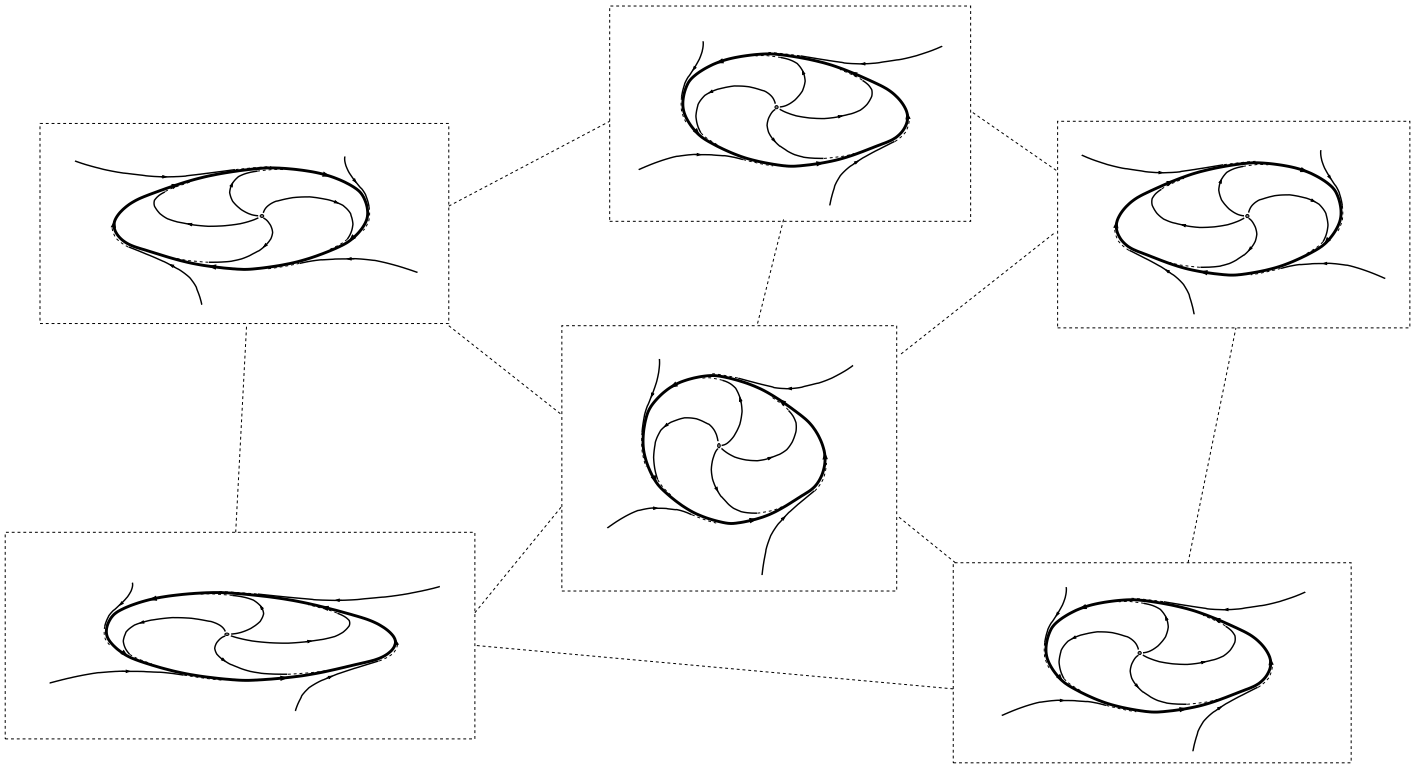
$$\dot{\theta}_t = f(\theta_t)\,.$$

Phase reduction and Kuramoto model

N dynamical systems of $\mathbb{R}^n$ in interaction :

$$\dot{X}_t^k = F_k(X_t^k) + G_k(X_t^1, \ldots, X_t^N),$$

each isolated one $\dot{X}_t = F_k(X_t)$ admitting a limit cycle.



Phase reduction and Kuramoto model

Kuramoto's reduction

If the N limit cycles are nearly identical and the interactions are weak, the model is well approximated by the dynamics of one-dimensional phases

$$\dot{\theta}_j = \omega_j + \sum_{i=1}^{N} \Gamma_{ij}(\theta_j - \theta_i) \qquad i = 1 \dots N\,,$$

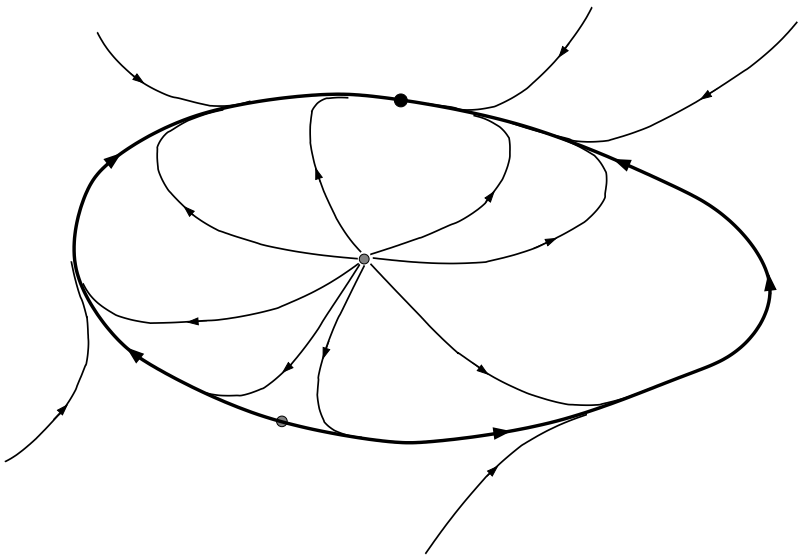
where ω_j is the natural oscillation frequency of the jth limit cycle, and the the interactions are given by the functions $\Gamma_{i,j}$.

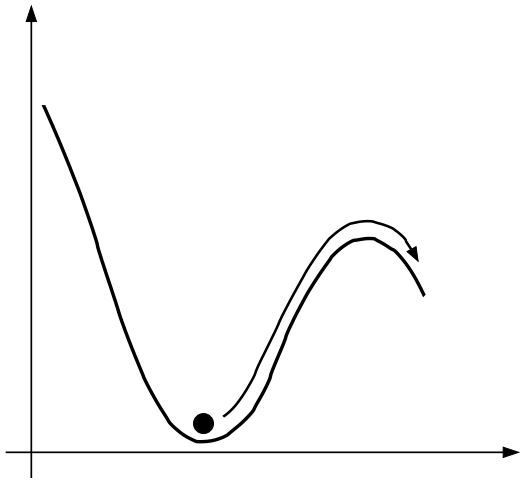
Phase reduction for models perturbed with noise

Consider a system in $\mathbb{R}^n$ of the type

$$\mathrm{d}X_t = F(X_t)\,\mathrm{d}t + \sqrt{\varepsilon}\,\mathrm{d}B_t$$

where B_t is a n-dimensional Brownian motion, and such that the non-perturbed system $\dot{X}_t = F(X_t)$ admits a stable curve M, on which the dynamics is slow.



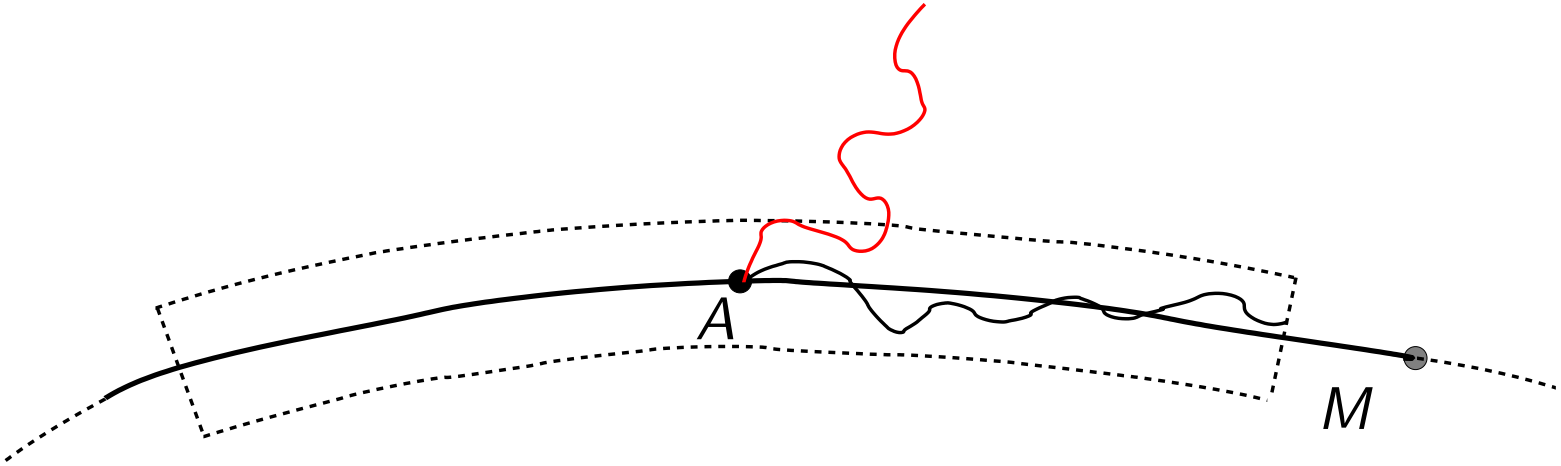


Escape problem for the model reduced on M

Phase reduction for models perturbed with noise

Validity of phase reduction if :

- The escapes from a fixed point A of M occur with high probability close to M.



- The probability of these escape paths can be well approximated by studying the escape problem for the model restraint on M.

Large deviations

Let P_x^ε be the law (on the space $C([T,0],\mathbb{R}^n)$, $-\infty < T < 0$) of the process

$$\mathrm{d}X_t \,=\, F(X_t)\,\mathrm{d}t + \sqrt{\varepsilon}\,\mathrm{d}B_t$$

starting at x. The family $(P_x^\varepsilon)_{\varepsilon>0}$ satisfies a large deviation principle of speed ε and rate function

$$I_T^x(Y) \,=\, \begin{cases} \frac{1}{2}\int_T^0 \left\| \dot{Y}_t - F(Y_t) \right\|^2 \mathrm{d}t & \text{if } Y \text{ is absolutely continuous} \\ & \text{and } Y_T = x\,, \\ +\infty & \text{otherwise}\,. \end{cases}$$

It means that for ε small

$$P_x^\varepsilon(O) \,\approx\, \exp\left(-\frac{1}{\varepsilon}\inf_{Y\in O} I_T^x(Y)\right)\,.$$

Quasipotential

For a connected compact $K \subset \mathbb{R}^n$, and two points A and B of K the quasipotential is defined by

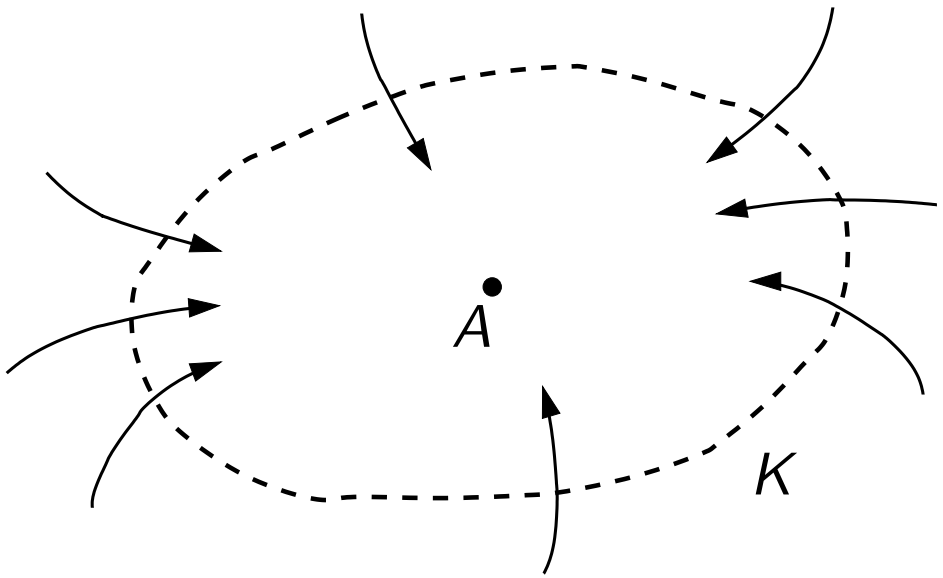
$$W_K(A,B) = \inf \Big\{ I_T^A(Y) : \, Y \in C([T,0],K), \, -\infty < T < 0, \\ Y_T = A, \, Y_0 = B \Big\}.$$

It represents the cost of reaching B from A with respect to the rate function I_T.

Escape problem : the Freidlin-Wentzell theory

$$\mathrm{d}X_t = F(X_t)\,\mathrm{d}t + \sqrt{\varepsilon}\,\mathrm{d}B_t$$

A fixed point, K compact smooth neighborhood included in the domain of attraction of A. Suppose moreover F Lipschitz.



Escape problem : the Freidlin-Wentzell theory

The first exit from K occurs for $\varepsilon \to 0$ with high probability very close to the points $B \in \partial K$ satisfying $W_K(A, B) = \inf_{E \in \partial K} W_K(A, E)$.

The time of first escape τ^ε satisfies

$$\mathbb{E}_A(\tau^\varepsilon) \approx \exp\left(\frac{1}{\varepsilon} W_K(A, B)\right) ,$$

and

$$\tau^\varepsilon / \mathbb{E}_A(\tau^\varepsilon) \to \mathcal{E}(1) .$$

Quasipotential : reversible case

If the system is reversible ($F = -\nabla V$), and if the solution of the deterministic system $\dot{X}_t = -\nabla V(X_t)$ with initial condition B converges to A for $t \to \infty$, then

$$W_K(A, B) \, = \, 2(V(B) - V(A)) \, .$$

Indeed :

$$\begin{aligned} I_T^A(Y) \, &= \, \frac{1}{2} \int_T^0 \left\| \dot{Y}_t + \nabla V(Y_t) \right\|^2 \mathrm{d}t \\ &= \, \frac{1}{2} \int_T^0 \left\| \dot{Y}_t - \nabla V(Y_t) \right\|^2 \mathrm{d}t + 2 \int_T^0 \nabla V(Y_t) \dot{Y}_t \, \mathrm{d}t \\ &= \, \frac{1}{2} \int_T^0 \left\| \dot{Y}_t - \nabla V(Y_t) \right\|^2 \mathrm{d}t + 2(V(B) - V(A)) \, . \end{aligned}$$

Stable fixed point and infinite time paths

If A is a stable fixed point and F Lipschitz

$$W_K(A,B) = \inf\left\{ I^A_{-\infty}(Y) : \; Y \in C((\infty,0],K)\,, \lim_{t\to-\infty} Y_t = A,\; Y_0 = B \right\}.$$

We call an optimal path an element of $C((\infty,0],K)$ that realizes the infimum.

Optimal paths and Euler Lagrange equation

Euler Lagrange equation

If F is C^2 and Y is an optimal path, then for all $t_1 < t_2$ such that $Y_t \in \mathring{K}$ for all $t \in (t_1, t_2)$, $Y \in C^2((t_1, t_2), \mathbb{R}^n)$ and

$$\ddot{Y}_t = DF^\dagger(Y_t)F(Y_t) + (DF(Y_t) - DF^\dagger(Y_t))\dot{Y}_t\,,$$

where † denotes the transposition operator.

Idea of proof

If Y is an optimal path, it is the optimal way to link Y_{t_1} to Y_{t_2}, and for f smooth with $f(t_1) = f(t_2) = 0$ and $\eta \in \mathbb{R}$ small

$$\frac{1}{2}\int_{t_1}^{t_2} \|\dot{Y}_t + \eta \dot{f}_t - F(Y_t + \eta f_t)\|^2 \,\mathrm{d}t - \frac{1}{2}\int_{t_1}^{t_2} \|\dot{Y}_t - F(Y_t)\|^2 \,\mathrm{d}t \geqslant 0\,.$$

After expansion the left hand side becomes

$$\eta \int_{t_1}^{t_2} \Big(\langle \dot{Y}_t, \dot{f}_t \rangle + \langle DF(Y_t) f_t, F(Y_t) \rangle - \langle \dot{f}_t, F(Y_t) \rangle \\ - \langle \dot{Y}_t, DF(Y_t) f_t \rangle \Big) \,\mathrm{d}t + O(\eta^2)$$

and the term of order η can be reformulated as

$$\int_{t_1}^{t_2} \Big\langle f_t, -\ddot{Y}_t + DF^\dagger(Y_t) F(Y_t) + DF(Y_t)\dot{Y}_t - DF^\dagger(Y_t)\dot{Y}_t \Big\rangle \,\mathrm{d}t\,.$$

A non reversible example

Model proposed by Maier and Stein (1995) :

$$F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x - x^3 - (1+\delta)xy^2 \\ -(1+x^2)y \end{pmatrix} = -\nabla V\begin{pmatrix} x \\ y \end{pmatrix} - \delta\begin{pmatrix} xy^2 \\ 0 \end{pmatrix}$$

with $V(x,y) = -\frac{1}{2}x^2 + \frac{1}{4}x^4 + \frac{1}{2}(1+x^2)y^2$.
$\{y=0\}$ is stable, $(0,0)$ is a sell point, $(1,0)$ is a stable fixed point.

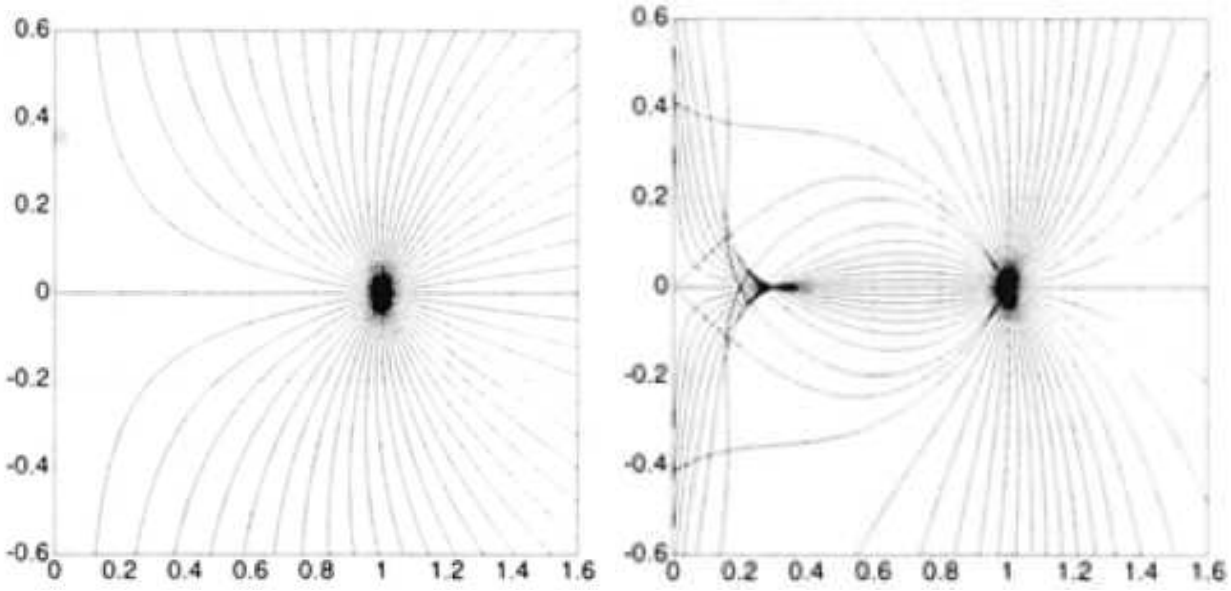


FIGURE: Optimal escape paths from $(1,0)$ for $\delta = 0$ and $\delta = 4$

Perturbation of reversible models

We consider models of the type

$$\mathrm{d}X_t \;=\; -\nabla V(X_t)\,\mathrm{d}t + \delta G(X_t)\,\mathrm{d}t + \sqrt{\varepsilon}\,\mathrm{d}B_t\,,$$

where the deterministic reversible part $\dot{X}_t = -\nabla V(X_t)$ admits a stable curve M^0 of stationary solutions.

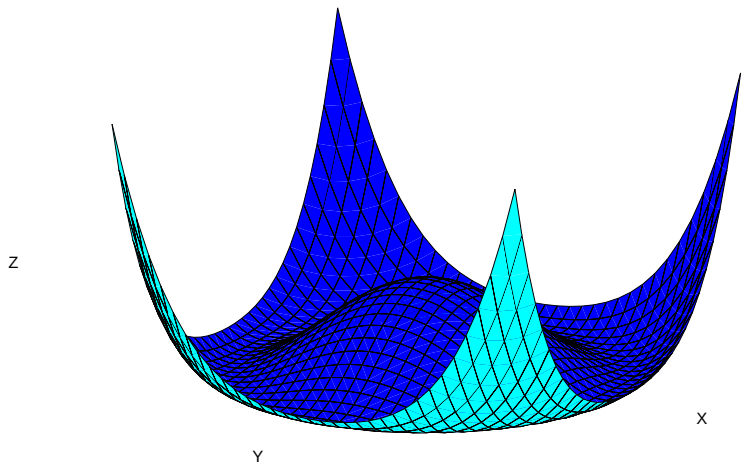


FIGURE: Example of such potential V in $\mathbb{R}^2$.

Link with the "Active Rotators"

The limit $N \to \infty$ of the "Active Rotators" model (each $\varphi_t^{j,N}$ represents a position on the circle $\mathbb{R}/2\pi\mathbb{R}$)

$$\mathrm{d}\varphi_t^{j,N} = \delta b(\varphi_t^{j,N})\,\mathrm{d}t + \sum_{i=1}^{N} J(\varphi_t^{j,N} - \varphi_t^{i,N})\,\mathrm{d}t + \mathrm{d}B_t^j$$

is given by the PDE

$$\partial_t p_t = \frac{1}{2}\Delta p_t - \nabla(p_t J * p_t) - \delta\nabla(p_t b)\,,$$

whose reversible part $\partial_t p_t = \frac{1}{2}\Delta p_t - \nabla(p_t J * p_t)$ admits a stable curve of stationary solutions.

Perturbation of reversible models

$$\mathrm{d}X_t \,=\, -\nabla V(X_t)\,\mathrm{d}t + \delta G(X_t)\,\mathrm{d}t + \sqrt{\varepsilon}\,\mathrm{d}B_t\,,$$

and M^0 curve of stationary solutions for the deterministic model.
So for all $X \in M^0$

$$\nabla V(X) \,=\, 0\,.$$

Moreover we suppose that for all $X \in M$ and all vector w normal to M^0 at X

$$\langle H(X)w, w\rangle \,\geqslant\, \lambda\|w\|^2\,.$$

With this hypothesis M^0 is a stable normally hyperbolic manifold.

Normal Hyperbolicity

For a flow $\dot{X}_t = F(X_t)$ admiting an invariant curve M, define the linearized evolution semi-group $\Phi(Q,t)$ following a trajectory $Q_t \in M$:

$$\Phi(Q,0)u \;=\; u \quad \text{for all}\, u \in \mathbb{R}^n \qquad \text{and} \quad \partial_t \Phi(Q,t) \;=\; DF(Q_t)\Phi(Q,t)\,.$$

M is normally hyperbolic if $\sup_{Q\in M} \nu(Q) < 1$ and $\sup_{Q\in M} \sigma(Q) < 1$, where ($P^N_Q$ and P^T_Q are the normal and tangent projections of M at Q)

$$\nu(Q) \;:=\; \inf\left\{ a : \left(\frac{\|w\|}{\|P^N_{Q_t}\Phi(Q,t)w\|} \right) a^t \to 0 \right.$$

$$\left. \text{as} \quad t \downarrow -\infty \quad \forall w \in N_Q \right\}$$

$$\sigma(Q) \;:=\; \inf\left\{ b : \frac{\|w\|^b / \|v\|}{\|P^N_{Q_t}\Phi(Q,t)w\|^b / \|P^T_{Q_t}\Phi(Q,t)v\|} \to 0 \right.$$

$$\left. \text{as} \quad t \downarrow -\infty \quad \forall v \in T_Q,\, w \in N_Q \right\}.$$

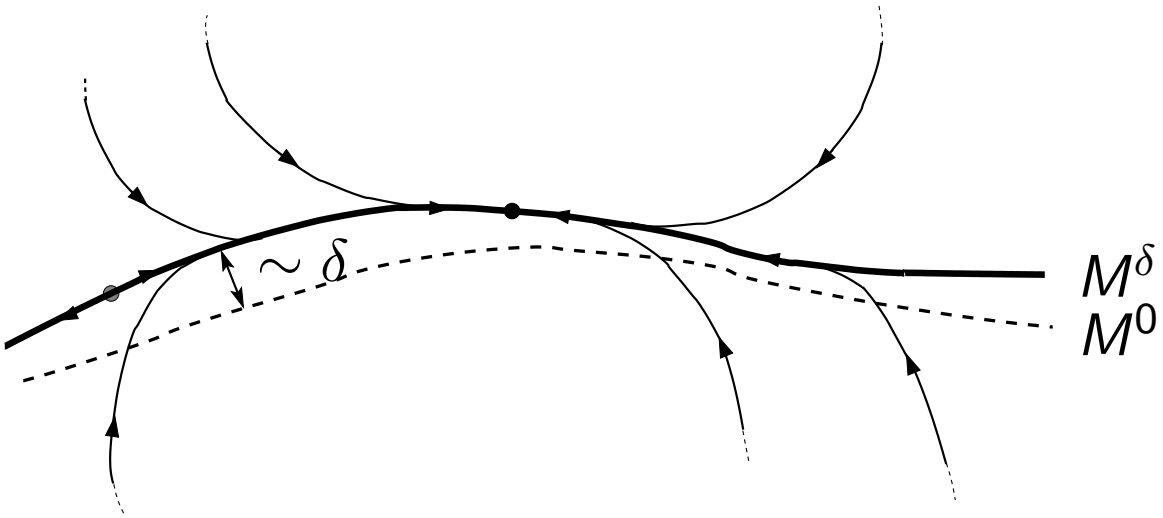
Persistence of the stable curve

Theorem [Fénichel, 1972]

The deterministic perturbed system

$$\dot{X}_t = -\nabla V(X_t) + \delta G(X_t)$$

admits a stable normally hyperbolic curve M^δ. Moreover $\mathrm{dist}(M^0, M^\delta) = O(\delta)$.



Dynamics on M^δ

For $\delta \geqslant 0$ consider a parametrization $q_\delta(\varphi)$ of M^δ satisfying $\|q'_\delta(\varphi)\| = 1$.

The deterministic evolution $\dot{X}_t = -\nabla V(X_t) + \delta G(X_t)$ reduced on M^δ is given for $\delta > 0$ by

$$\dot{\varphi}_t \;=\; b_\delta(\varphi_t)\,,$$

where

$$b_\delta(\varphi) \;:=\; \Big\langle -\nabla V[q_\delta(\varphi)] + \delta G[q_\delta(\varphi)], q'_\delta(\varphi) \Big\rangle\,.$$

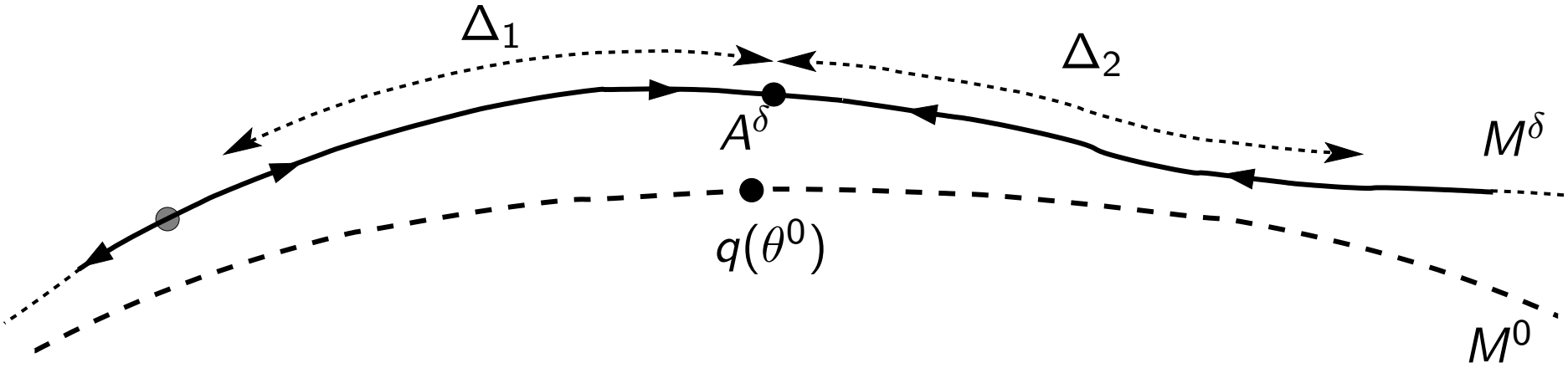
Dynamics on M^δ

$b_\delta \to 0$ (M^0 is a curve of stationary solutions), and $b_\delta/\delta \to b_0$, with

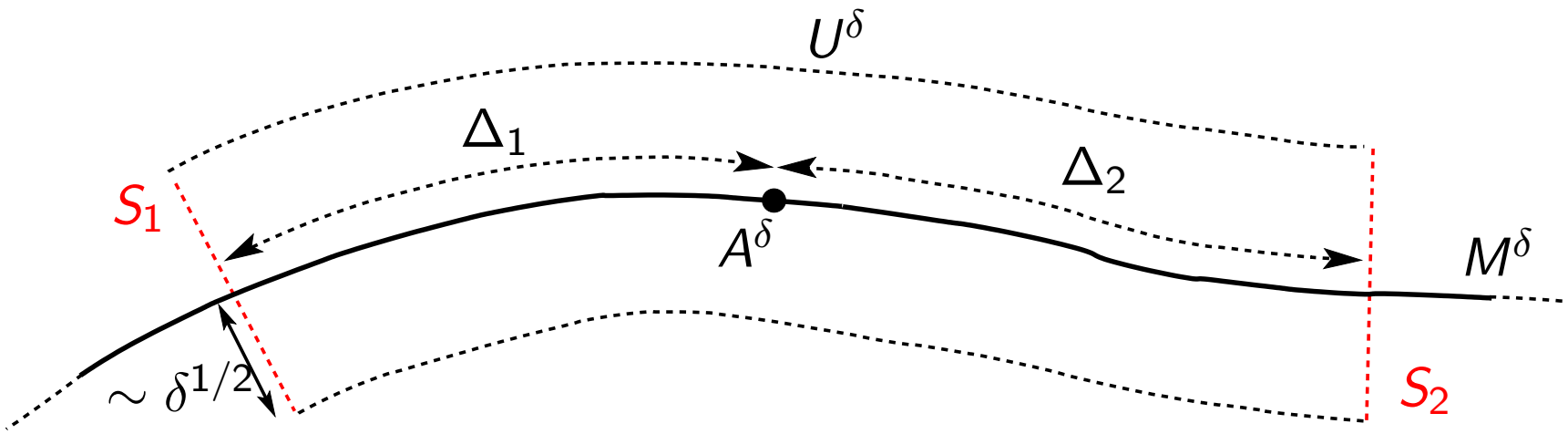
$$b_0(\theta) := \langle G[q_0(\theta)], q_0'(\theta) \rangle .$$

Suppose that the system $\dot{\theta} = b_0(\theta)$ admits a fixed point θ^0 with $b_0'(\theta^0) < 0$ and $[\theta^0 - \Delta_1, \theta^0 + \Delta_2]$ is included in its domain of attraction

Then for all δ small enough, $\dot{\varphi}_t = b_\delta(\varphi_t)$ admits also a fixed point φ_{A^δ}, with $[\varphi_{A^\delta} - \Delta_1, \varphi_{A^\delta} + \Delta_2]$ included in its domain of attraction.



Consider the tube U^δ as follows :



Define W_δ the quasipotential associated to the compact U^δ and the process

$$\mathrm{d}X_t \;=\; -\nabla V(X_t)\,\mathrm{d}t + \delta G(X_t)\,\mathrm{d}t + \sqrt{\varepsilon}\,\mathrm{d}B_t\,,$$

W_δ^{red} the (1-dimensional) quasipotential associated to the compact $[\varphi_{A^\delta} - \Delta_1, \varphi_{A^\delta} + \Delta_2]$ and the one-dimensional process

$$\mathrm{d}\varphi_t \;=\; b_\delta(\varphi_t)\,\mathrm{d}t + \sqrt{\varepsilon}\,\mathrm{d}B_t^1\,.$$

In fact, for $\varphi \in [\varphi_{A^\delta} - \Delta_1, \varphi_{A^\delta} + \Delta_2]$, $W_\delta^{red}(\varphi_{A^\delta}, \varphi) \;=\; 2\int_{\varphi_{A^\delta}}^{\varphi} b_\delta(\varphi')\,\mathrm{d}\varphi'\,.$

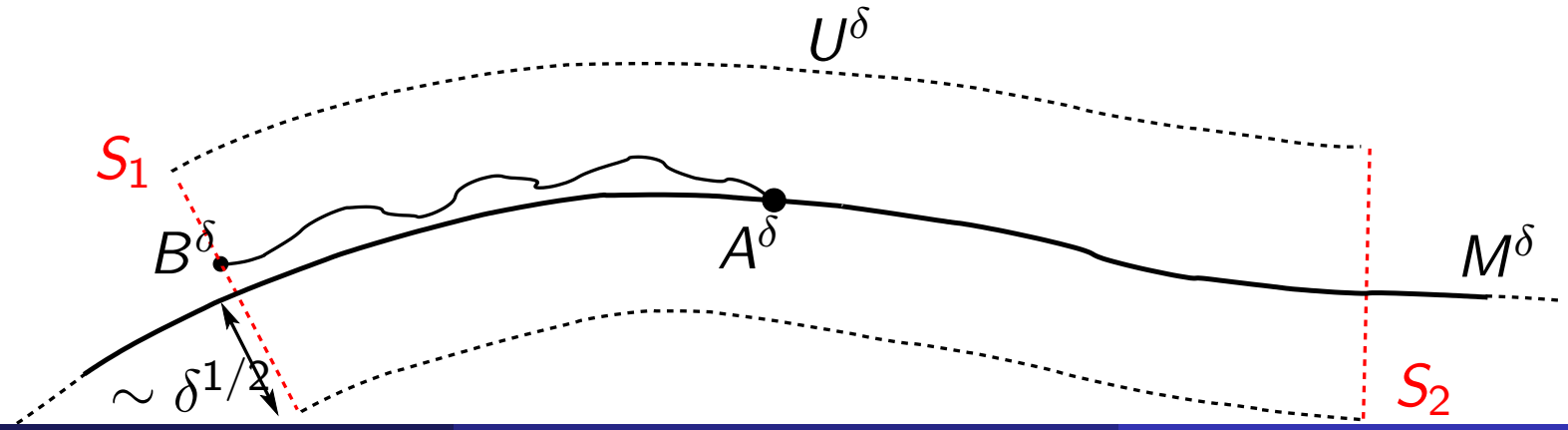
Theorem (P. 2013)

For δ small enough, if $B^\delta \in \partial U^\delta$ satisfies

$$W_\delta\left(A^\delta, B^\delta\right) = \inf_{E \in \partial U^\delta} W_\delta\left(A^\delta, E\right),$$

then $B^\delta \in S_1 \cup S_2$ and

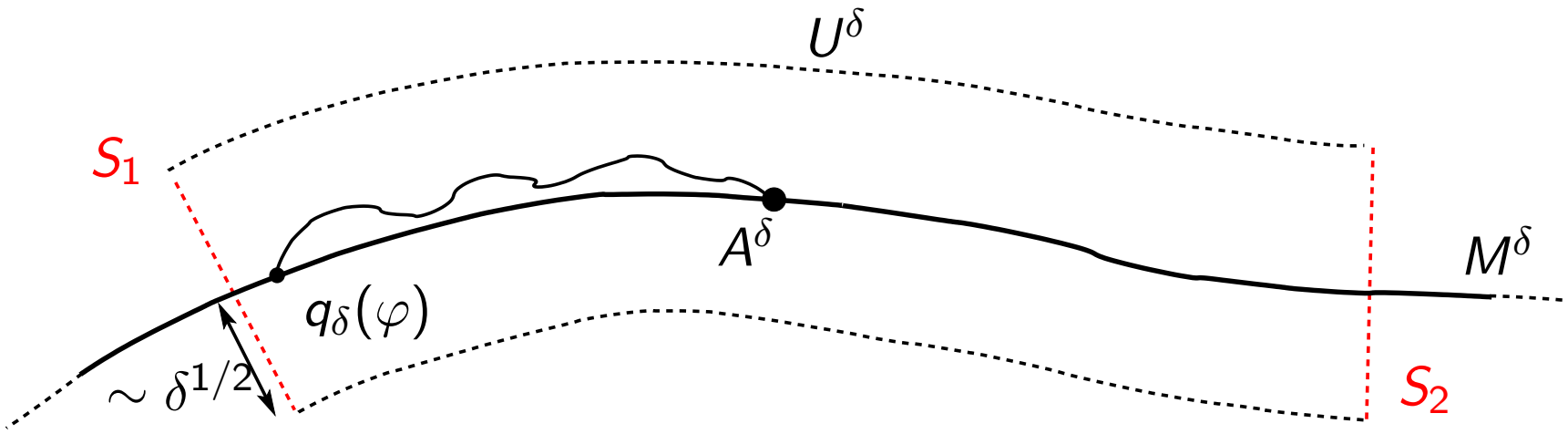
$$\begin{aligned} W_\delta\left(A^\delta, B^\delta\right) &= W_\delta^{red}\left(\varphi_{A^\delta}, \varphi_{B^\delta}\right) + O\left(\delta^3 |\log \delta|^3\right) \\ &= 2\int_{\varphi_{A^\delta}}^{\varphi_{B^\delta}} b_\delta(\varphi)\,\mathrm{d}\varphi + O\left(\delta^3 |\log \delta|^3\right). \end{aligned}$$



Corollary

For δ small enough and $\varphi \in [\varphi_{A^\delta} - \Delta_1, \varphi_{A^\delta} + \Delta_2]$ we have

$$W_\delta\Big(A^\delta, q_\delta(\varphi)\Big) = W_\delta^{red}\Big(\varphi_{A^\delta}, \varphi\Big) + O\left(\delta^3 |\log \delta|^3\right)$$
$$= 2\int_{\varphi_{A^\delta}}^{\varphi} b_\delta(\varphi')\,\mathrm{d}\varphi' + O\left(\delta^3 |\log \delta|^3\right) .$$



Idea of proof

- $b_\delta = O(\delta)$, and thus $W_\delta^{red} = O(\delta)$.
- The reversible part of $W_\delta(A^\delta, E)$ is $2(V(E) - V(A^\delta))$.
  Since for E close to M^0 we have

$$V(E) - V(M^0) \sim \lambda \, \mathrm{dist}(E, M^0)^2 \,,$$

  for E such that $\mathrm{dist}(E, M) = O(\delta^{1/2})$ we have

$$V(E) - V(A) = V(E) - V(M^0) - (V(A^\delta) - V(M^0)) \sim \lambda\delta + O(\delta^2) \,.$$

Since $\mathrm{dist}(M, M^\delta) = O(\delta)$ going at a distance $C\delta^{1/2}$ from M^δ with C large is more expensive than exiting the tube by staying on M^δ.

Idea of proof

If $G = -\nabla g$, the solutions of the Euler-Lagrange equation are the deteministic solution of $\dot{X}_t = -\nabla V(X_t) + \delta G(X_t)$ time reversed.

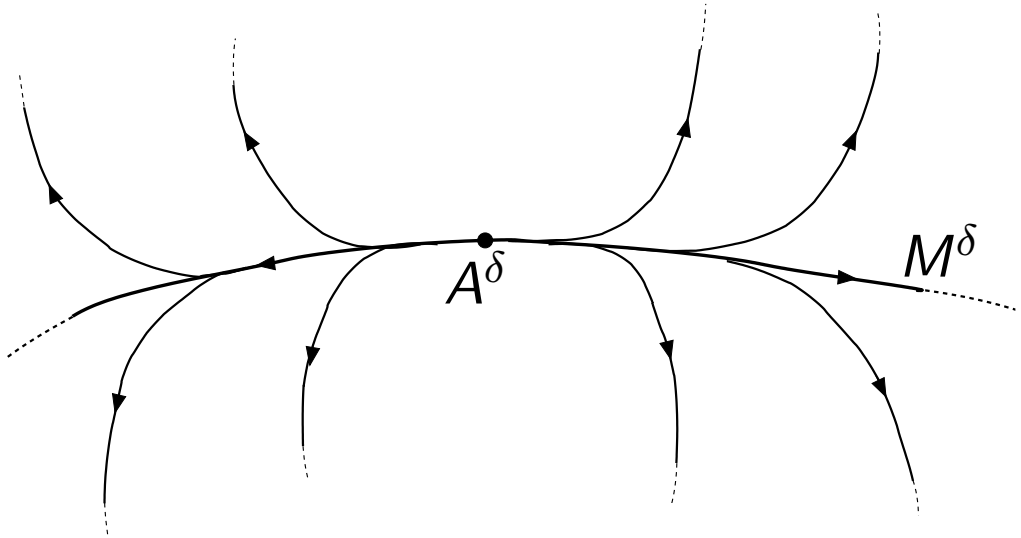


FIGURE: Solutions of E-L if $G = -\nabla g$.

Idea of proof

Euler-Lagrange equation :

$$\ddot{Y}_t = \Big(H(Y_t) - \delta DG^\dagger(Y_t)\Big)\Big(\nabla V(Y_t) - \delta G(Y_t)\Big) + \delta\Big(DG(Y_t) - DG^\dagger(Y_t)\Big)\dot{Y}_t\,.$$

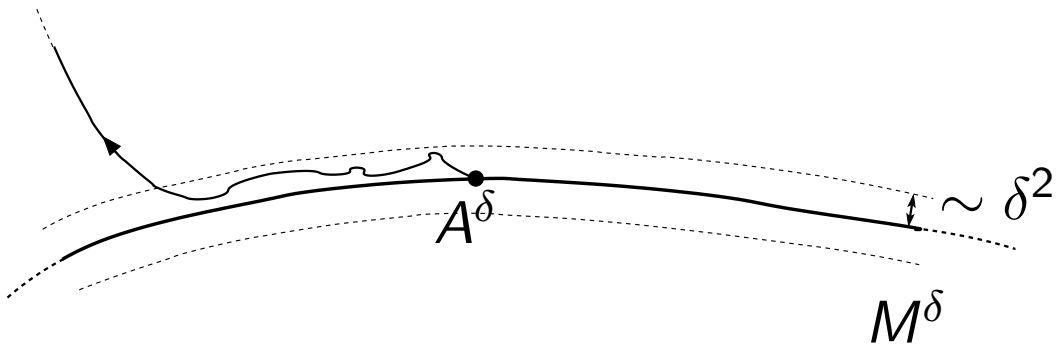


FIGURE: Shape of a solution of E-L if G is not a gradient.