Biased random walk among random conductances: Einstein relation and monotonicity of the speed

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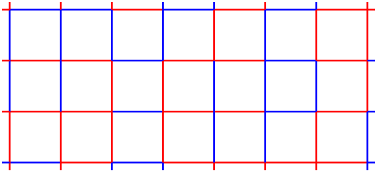
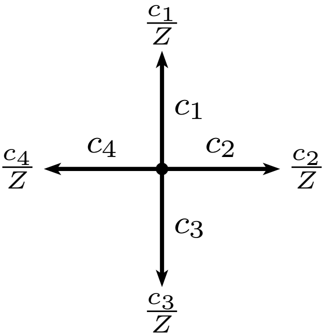
joint work with Jan Nagel and Xiaoqin Guo

Outline

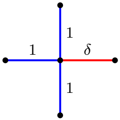
The Random Conductance Model

We define a random medium by giving random weights - often called "conductances" - to the bonds of the lattice.
Consider first the case where the weights are independent, with the same law. Assume that they are bounded above and bounded away from zero.
The configurations of the weights is called "environment". For a fixed environment, define the law of a random walk, where the transition probabilities from a point to its neighbours are proportional to the weights of the bonds.





where $Z := c_1 + c_2 + c_3 + c_4$.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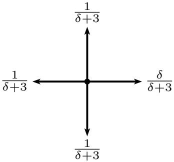
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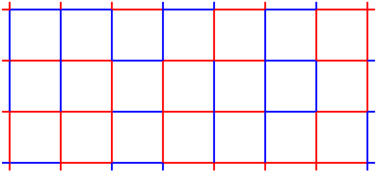

General question:

Question

Can the random medium be replaced by an "averaged" deterministic medium?

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There are two, contradicting paradigms in the theory of random media: *Homogenization* versus *Intermittency*.

More precisely, consider

- Nearest-neighbour random walk in $\mathbb{Z}^d$
- Random conductances $\omega = \{\omega_{x,y}\}_{x \sim y}$ with law P such that
  (i) $\{\omega_{x,y}\}_{x \sim y}$ are iid

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- Nearest-neighbour random walk in $\mathbb{Z}^d$
- Random conductances $\omega = \{\omega_{x,y}\}_{x\sim y}$ with law P such that
    - (i) $\{\omega_{x,y}\}_{x\sim y}$ are iid
    - (ii) $\frac{1}{\kappa} \leq \omega_{x,y} \leq \kappa \quad \forall\ x \sim y$ (uniform ellipticity)

For fixed ω, the RW in the environment ω is the Markov chain given by

$$P_\omega^x(X_{n+1} = x + e | X_n = x) = \frac{\omega_{x,x+e}}{\sum_{|e'|=1} \omega_{x,x+e'}} \qquad \text{(quenched law)}$$

Average over environments: $\mathbb{P}^x = \int P_\omega^x(\cdot) P(d\omega)$ (averaged law)

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Answer: Yes! $\frac{1}{\sqrt{n}} X_{\lfloor n \cdot \rfloor} \xrightarrow[n\to\infty]{d}$ Brownian motion with covariance $\Sigma = \sigma \cdot \mathrm{Id}$ (under P_ω). There are several papers by A. de Masi/P. A. Ferrari/S. Goldstein/W. D. Wick, D. Boivin, L. Fontes/P. Mathieu, S. M. Kozlov, V. Sidoravicius/A.-S. Sznitman,... leading to this result, and it has been extended to the case of bounded, strictly positive conductances.

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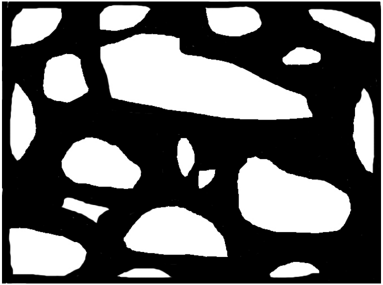
An active direction of research is the extension of this theorem to the case when the conductances form a stationary, ergodic random field. It is not true in general, but true under boundedness conditions on the conductances. In this case, the covariance matrix Σ is not diagonal in general.

Question

How does Σ depend on the law of the conductances?

Note that this is important from the viewpoint of "material sciences"!

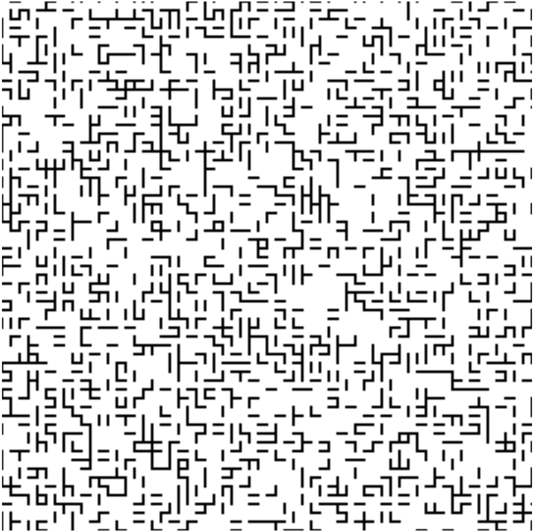
Composite Material



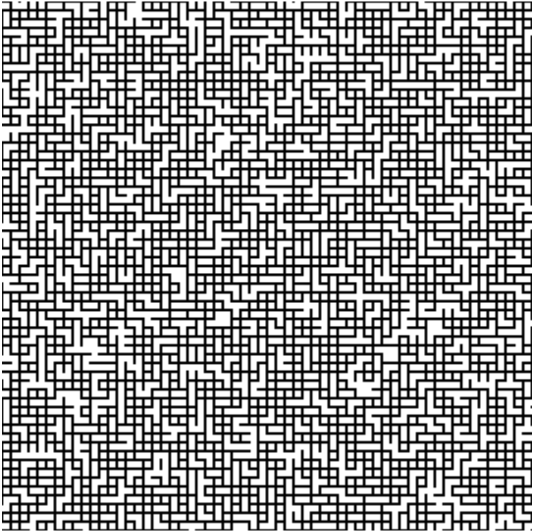
Random walks on supercritical percolation clusters

To be more radical, consider bond percolation with parameter p on the d-dimensional lattice: all bonds are *open* with probability p and *closed* with probability $1-p$, independently of each other. This corresponds to conductances with values either 1 or 0.

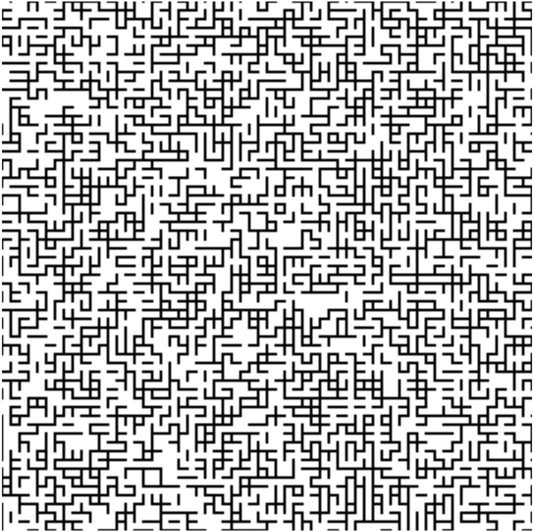
Bond percolation p= 0.25



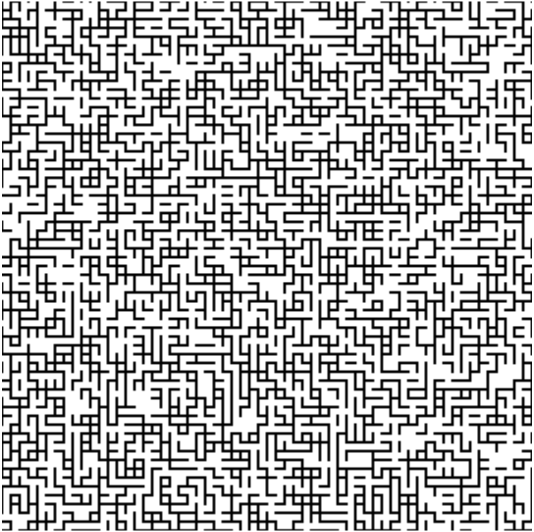
Bond percolation p= 0.75



Bond percolation p= 0.48



Bond percolation p= 0.51



Note: this model shows a *phase transition* in p. Assume $d \geq 2$.
Then, there is a critical value $p_c = p_c(d) \in (0, 1)$ such that the probability that the origin is in an infinite connected component is strictly positive for $p > p_c$ and zero for $p < p_c$.

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In fact there is, with probability 1, at most one infinite connected component. It is called "infinite cluster".

Take bond percolation on $\mathbb{Z}^d$, $d \geq 2$. Choose p close enough to 1 such that there is a (unique) infinite cluster.
Condition on the event that the origin is in the infinite cluster.
Start a random walk in the infinite cluster which can only walk on open bonds, and which goes with equal probabilities to all neighbours. (In particular, this random walk never leaves the infinite cluster.)

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Answer: Yes! (This was proved by Noam Berger/Marek Biskup, Pierre Mathieu/Andrey Piatnitski, Vladas Sidoravicius/Alain-Sol Sznitman). Method of proof: decompose the walk in a martingale part and a "corrector". Show that the corrector can be neglected and apply the CLT for martingales.

Einstein-Relation

The Einstein relation gives a different interpretation of the variance as the derivative of the speed of the random walk, when one has a drift in a "favourite" direction ℓ. This leads us to consider **biased random walks in random environments**.

Biased random walks among random conductances

- RW with bias of strength $\lambda > 0$:

$$P^x_{\omega,\lambda}(X_{n+1} = x + e | X_n = x) = \frac{\omega_{x,x+e}\, e^{\lambda \ell \cdot e}}{\sum_{|e'|=1} \omega_{x,x+e'}\, e^{\lambda \ell \cdot e'}}$$

- $\mathbb{P}^x_\lambda = \int P^x_{\omega,\lambda}(\cdot) P(d\omega)$

For the random conductance model, we have:

Theorem

(Lian Shen 2002)
For fixed drift, there is a law of large numbers:
For any $\lambda > 0$, $\frac{1}{n} X_n \xrightarrow[n\to\infty]{} v(\lambda)$ $\mathbb{P}_\lambda^x$-*a.s. where* $v(\lambda)$ *is deterministic and* $v(\lambda) \cdot \ell > 0$.

For the RW among random conductances, the Einstein relation holds.

Theorem

Einstein-Relation (NG, Jan Nagel und Xiaoqin Guo, to be posted soon)
Assume that the conductances are iid and uniformly elliptic. Then,

$$\lim_{\lambda\to 0}\frac{v(\lambda)}{\lambda}=\Sigma\ell\,.$$

Further, $v(\lambda)$ is differentiable for all λ and we can write its derivative as a covariance.

The theorem has been proved by Tomasz Komorowski and Stefano Olla (2005) in the case where $d \geq 3$ and the conductances only take two values.

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It can be proved easily in the one-dimensional case and in the periodic case.

The Einstein relation for the random conductance model is a consequence of the following expansion. Let Q_λ denote the invariant measure for the process $\bar{\omega} = (\bar{\omega}_n)_{n\geq 0}$, the *environment seen from the particle*, where $\bar{\omega}_n = \theta_{X_n}\omega$ and the law of $(X_n)_n$ is given by $P_{\omega,\lambda}$.
Write $\mathcal{F}$ for the set of bounded functions $f : \Omega \to \mathbb{R}$ depending only on a finite set of conductances. We show the following first order expansion of Q_λ around $\lambda = 0$.

Theorem

Let $d \geq 3$.
There exists a functional Λ on $\mathcal{F}$, such that

$$\lim_{\lambda\to 0} \frac{Q_\lambda f - Q_0 f}{\lambda} = \Lambda f$$

for any $f \in \mathcal{F}$.

More precisely, Λ is given as follows. Let $d(\omega, x) = E_\omega^x[(X_1 - X_0)]$, consider the 2-dimensional process

$$\frac{1}{\sqrt{n}}\left(\sum_{k=1}^{n} f(\bar{\omega}_k) - Q_0 f, \sum_{k=1}^{n} d(\bar{\omega}_k, 0)\cdot \ell\right),$$

By the Kipnis-Varadhan Theorem, this process converges in distribution under $\mathbb{P}$ to a 2-dimensional Gaussian random variable (N_f, N_d). Then $\Lambda f = -Cov(N_f, N_d)$.

To see how the Einstein relation follows, note that, defining $\mathbb{Q}_\lambda = Q_\lambda \otimes P_{\omega,\lambda}$, by the ergodic theorem we can write the velocity as

$$v(\lambda) = \lim_{n\to\infty} \frac{X_n}{n} = \mathbb{E}_{\mathbb{Q}_\lambda}[d(\omega^\lambda, 0)]$$

where the limit is almost surely under $\mathbb{Q}_\lambda$ and

$$d(\omega^\lambda, 0) = \sum_{|e|=1} (C_0^\lambda)^{-1} \omega_{0,e} e^{\lambda\ell\cdot e} e,$$

with $C_0^\lambda = \sum_{|e'|=1} \omega_{0,e'} e^{\lambda\ell\cdot e'}$. Write

$$d(\omega^\lambda, 0) = d(\omega, 0) + A(\omega) + o(\lambda)$$

Consequently, we can apply the expansion to (the components of) the drift $d(\cdot, 0) \in \mathcal{F}$:

$$\lim_{\lambda \to 0} \frac{v(\lambda)}{\lambda} = \lim_{\lambda \to 0} \frac{Q_\lambda d(\omega^\lambda, 0) - Q_0 d(\omega, 0)}{\lambda}$$

$$= \lim_{\lambda \to 0} \left(\frac{Q_\lambda d(\omega, 0) - Q_0 d(\omega, 0)}{\lambda} + Q_\lambda[A] \right)$$

$$= \Lambda d(\cdot, 0) + Q_0[A].$$

and one can evaluate the two terms.

To prove the expansion for Q_λ, we show

Theorem

Diffusivity part

For any $t \geq 1$ and $f \in \mathcal{F}$, we have

$$\lim_{\lambda \to 0} \frac{\frac{\lambda^2}{t} \mathbb{E}_{Q,\lambda} \sum_{k=0}^{t/\lambda^2} f(\bar{\omega}_k) - Q_0 f}{\lambda} = \Lambda f,$$

where $\mathbb{E}_{Q,\lambda}$ is the expectation with respect to $Q_0 \otimes P_{\omega^\lambda}$.

Theorem

Mobility part

There exists a constant C depending only on the dimension and the ellipticity constants, such that for any $t \geq 1$ and $f \in \mathcal{F}$

$$\frac{\frac{\lambda^2}{t} \mathbb{E}_{Q,\lambda} \sum_{k=0}^{t/\lambda^2} f(\bar{\omega}_k) - Q_\lambda f}{\lambda} \leq \frac{C}{\sqrt{t}}.$$

Before going to the proof of the diffusivity part, note that Joel Lebowitz and Hermann Rost showed, using the invariance principle and Girsanov transform:

Theorem

(Joel Lebowitz, Hermann Rost, 1994) Let $\alpha > 0$*. Then*

$$\lim_{\lambda\to 0\,,t\to+\infty\,,\lambda^2 t=\alpha} \mathbb{E}_0\left[\frac{X^\lambda(t)}{\lambda t}\right] = \Sigma\,\ell\,.$$

Proof of the Diffusivity part

- Consider first term $\frac{\lambda}{t}\sum_{k=0}^{t/\lambda^2}(f(\bar{\omega}_k) - Q_0 f)$ and assume without loss of generality $Q_0 f = 0$. From Kipnis-Varadhan, have a decomposition

$$\sum_{k=0}^{n} f(\bar{\omega}_k) = M_n^* + R_n,$$

where M_n^* is a martingale under $\mathbb{Q}_0$ and $\frac{R_n}{\sqrt{n}}$ converges in law to 0.

- With $d_\ell(\omega, x) = E_\omega^x[(X_1 - X_0)\cdot \ell]$ the expected displacement in direction ℓ, apply martingale CLT to get the joint convergence

$$\lambda\left(\sum_{k=0}^{t/\lambda^2} f(\bar{\omega}_k),\ \ell\cdot X_{t/\lambda^2} - \sum_{i=1}^{t/\lambda^2} d_\ell(\omega, X_{i-1})\right) \xrightarrow[\lambda\to 0]{} (N_t^*, N_t)$$

in distribution under $\mathbb{Q}_0$ to some 2-dimensional Brownian motion.

- Show convergence of density: under P_ω and in L^p:

$$\log \frac{dP_{\omega,\lambda}}{dP_\omega}(X_s)_{0\le s\le (t/\lambda^2)} \xrightarrow[\lambda\to 0]{} N_t - \frac{1}{2}E[N_t^2]$$

where (N_t) is a Brownian motion.

- Prove boundedness statements to conclude convergence of the expectation

$$\mathbb{E}_{Q,\lambda}\left[\frac{\lambda}{t}\sum_{k=0}^{t/\lambda^2} f(\bar{\omega}_k)\right] = \mathbb{E}_{Q,0}\left[\frac{\lambda}{t}\sum_{k=0}^{t/\lambda^2} f(\bar{\omega}_k)\frac{dP_{\omega,\lambda}}{dP_\omega}(X_s)_{0\le s\le (t/\lambda^2)}\right]$$

Hence

$$\mathbb{E}_{Q,\lambda}\left[\frac{\lambda}{t}\sum_{k=0}^{t/\lambda^2} f(\bar{\omega}_k)\right] \xrightarrow[\lambda\to 0]{} \frac{1}{t}E\left[N_t^* e^{N_t - \frac{1}{2}E[N_t^2]}\right]$$

- Apply Girsanov's formula

$$E\left[N_t^* e^{N_t - \frac{1}{2}E[N_t^2]}\right] = E\left[[N^*, N]_t\right] = [N^*, N]_t,$$

- Remains to identify $[N^*, N]_t$. Note that the process $(\bar{\omega}_n)_{n\geq 0}$ is reversible under the law $\mathbb{Q}$. We can consider the processes $(X_n)_{n\geq 0}$, $\left(\sum_{i=0}^{n-1} d(\omega, X_i)\right)_{n\geq 1}$ and $\left(\sum_{k=0}^{n-1} f(\bar{\omega}_k)\right)_{n\geq 1}$ as functionals of the process $(\bar{\omega}_n)_{n\geq 0}$ under the law $\mathbb{Q}$. Indeed, the increments $X_m - X_n$ can a.s. be reconstructed from $\bar{\omega}_m$ and $\bar{\omega}_n$. More precisely, we view the family $(X_m - X_n)_{m,n}$ of increments as an additive functional, meaning that

$$(X_m - X_n) + (X_n - X_k) = (X_m - X_k)$$

which is antisymmetric with respect to time reversal,

$$(X_m - X_n) = -(X_n - X_m).$$

- Similarly, the family of increments $\left(\sum_{i=n}^{m-1} d(\omega, X_i)\right)_{m,n}$ is additive and symmetric, as is $\left(\sum_{k=n}^{m-1} f(\bar{\omega}_k)\right)_{m,n}$. Hence, the increments of the two processes are orthogonal,

$$E_{\mathbb{Q}}\left[(X_m - X_n)\cdot\left(\sum_{k=n}^{m-1} f(\bar{\omega}_k)\right)\right] = 0,$$

(this argument goes back to de Masi et al.)
Consequently, by the ergodic theorem and the invariance principle,

$$\mathrm{Cov}(N_t^*, N_t) = \lim_{n\to\infty}\frac{1}{n}\mathbb{E}_{\mathbb{Q}}\left[\left(\sum_{k=0}^{tn-1} f(\bar{\omega}_k)\right)\left(\ell\cdot X_{tn} - \sum_{i=0}^{tn-1}\ell\cdot d(\omega, X_i)\right)\right]$$

$$= -t\mathrm{Cov}\left(N_f, N_d\right) = t\Lambda f\,.$$

Proof of the Mobility part

We define a suitable regeneration structure. Need a-priori estimates. A lot of analytical effort goes here, and it is here that we need that the conductances are uniformly bounded and iid.

The Einstein relation is conjectured to hold for many models, but it is proved for few. Apart from the results mentioned, examples include:

- Random walks in balanced random environments (Xiaoqin Guo)
- Symmetric diffusions in random environment (NG, Pierre Mathieu, Andrey Piatnitski)
- Random walks on Galton-Watson trees (Gérard Ben Arous, Yueyun Hu, Stefano Olla, Ofer Zeitouni)
- Tagged particle in asymmetric exclusion (Michail Loulakis)

The following examples are in progress:

- Random walks on percolation clusters of ladder graphs (NG, Matthias Meiners, Sebastian Müller)
- Mott random walks (Alessandra Faggionato, NG, Michele Salvi)

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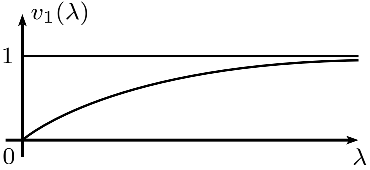
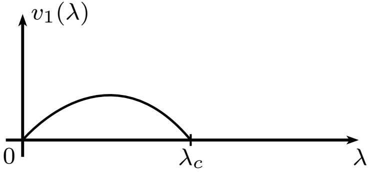
Question:
Is $v_1(\lambda) = v(\lambda) \cdot \ell$ increasing as a function of λ?

Back to the homogeneous medium: in this case, $v(\lambda)$ can be computed and $v_1(\lambda) = v(\lambda) \cdot \ell$ looks as follows:



For the speed of the random walk on an infinite percolation cluster, the following picture is conjectured:
for each $p \in (p_c, 1)$ we have, with $v_1(\lambda) = v(\lambda) \cdot \ell$



Reason for the zero speed regime: "traps" in the percolation cluster!
Alexander Fribergh and Alan Hammond showed recently that there is, for each $p \in (p_c, 1)$, a critical value λ_c such that $v_1(\lambda) > 0$ for $\lambda < \lambda_c$ and $v_1(\lambda) = 0$ for $\lambda > \lambda_c$. Quoting from their paper:

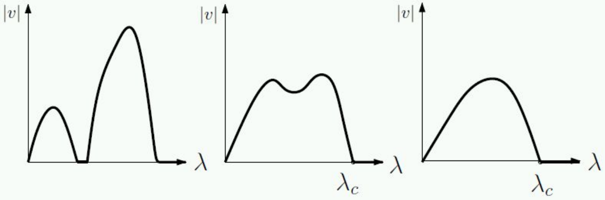
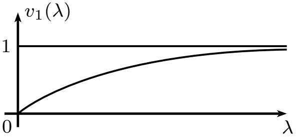


FIGURE 3. The speed as a function of the bias. Sznitman and Berger, Gantert and Peres established positive speed at low λ, but their works left open the possibility depicted in the first sketch. Our work rules this out, though the behavior of the speed in the ballistic regime depicted in the second sketch remains possible. The third sketch shows the unimodal form predicted physically.

How does $v_1(\lambda)$ depend on λ for the random walk among uniformly elliptic random conductances?

For the homogeneous medium, we have



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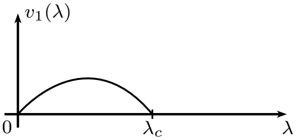
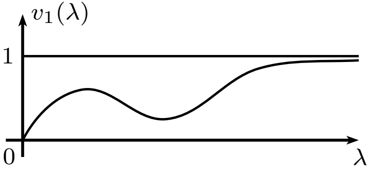


For the random walk among random conductances, we believe that the picture can be



We show (Noam Berger, NG, Jan Nagel, in progress): the speed in the favourite direction is in general *not* increasing. More precisely, assume the conductances take the values 1 (with probability $> p_c$) and δ with probability $1 - p$. Then, for δ small enough, there are $0 < \lambda < \lambda'$ such that $v_1(\lambda) > v_1(\lambda')$.

On the other hand, we show that the speed in the favourite direction *is* increasing, provided δ is close enough to 1 and the conductances take the values 1 (with probability p) and δ with probability $1 - p$.

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Hence, many new questions arise!

Thanks for your attention!