Ergodic theory for the symmetric inclusion process

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Framework

A successful coupling + sketch of the proof

The characterisation of the invariant measures

The big picture

Characterisation of the invariant measures.

An *attractive version* of the symmetric exclusion process (SEP), called **symmetric inclusion process (SIP)**.
Giardinà, C. - Kurchan, J. - Redig, F. and Vafayi, K.: JSP '09
Giardinà, C. - Redig, F. and Vafayi, K.: JSP '10

Liggett, T.M.: Interacting Particle Systems (1985, re-ed. 2005).
Rather complete study of the ergodic theory of the SEP: only extremal invariant measures are Bernoulli measures with constant density.

How-to (for the SEP):
- self-duality (invariant measures $\leftrightarrow$ bounded harmonic functions).
- successful coupling : bounded harmonic functions are constant.
- De Finetti's theorem : any invariant measure is convex combination of Bernoulli measures.

The setup

The SIP with parameter m on $\mathbb{Z}^d$: $\eta \in \mathbb{N}^{\mathbb{Z}^d}$.

Particles perform :

- simple symmetric nearest-neighbour random walks at rate $m/2$ and transition probability $p(\cdot,\cdot)$: for any $x, y \in \mathbb{Z}^d$

$$p(x,y) = \begin{cases} \dfrac{1}{2d} \text{ if } |x-y| = 1 \\ 0 \text{ otherwise} \end{cases},$$

- inclusion jumps: any particle invites a neighbouring particle to come over its location at rate 1.

$$\mathcal{L}f(\eta) = \sum_{x,y\in\mathbb{Z}^d} p(x,y)\eta(x)\Big(\frac{m}{2} + \eta(y)\Big)\Big[f(\eta^{x,y}) - f(\eta)\Big]$$

where $\eta^{x,y} = \eta - \delta_x + \delta_y$; here, δ_x denotes the configuration with a single particle at x and none elsewhere.

Some properties of the SIP

Giardinà, C. - Kurchan, J. - Redig, F. and Vafayi, K.: JSP '09

$$D(\xi,\eta)=\prod_{x\in\mathbb{Z}^d} d(\xi(x),\eta(x))$$

where $d(k,\ell)=\begin{cases} \dfrac{\ell!}{(\ell-k)!}\dfrac{\Gamma(\frac{m}{2})}{\Gamma(\frac{m}{2}+k)} \text{ if } k\le\ell, \\ 0 \text{ if } k>\ell. \end{cases}$

The SIP(m) is self-dual with respect to duality functions $D(\cdot,\cdot)$., i.e.

$$\mathbb{E}_\xi D(\xi_t,\eta)=\mathbb{E}_\eta D(\xi,\eta_t).$$

Note that ξ is a **finite SIP** (finitely many particles).
From now onwards, set $|\xi|=n$. And track down the locations of each particle by identifying ξ with a n-tuple $(X_1,...,X_n)$, so that

$$\xi=\sum_{j=1}^{n}\delta_{X_i}.$$

Some properties of the SIP

Giardinà, C. - Redig, F. and Vafayi, K.: JSP '10

Fix $\lambda \in [0, 1)$. Gamma product measures ν_λ^m,

$$\nu_\lambda^m\{\eta(x) = k)\} = \frac{1}{Z_{\lambda,m}} \frac{\lambda^k}{k!} \frac{\Gamma(\frac{m}{2} + k)}{\Gamma(\frac{m}{2})}, \; x \in \mathbb{Z}^d$$

are reversible and ergodic for the SIP (m).

Q: Are these the only one ?

Definition

Consider a probability measure μ. Denote its D-transform by :

$$\widehat{\mu}(\xi) := \int D(\xi, \eta) d\mu(\eta).$$

A probability measure μ is said to be *tempered* if: for all $n \in \mathbb{N}$,

$$c_n := \sup_{|\xi|=n} \int D(\xi, \eta) d\mu(\eta) < \infty$$

with the assumption c_n satisfies

$$\sum_{n \geq 1} c_n^{-1/n} = \infty$$

to guarantee the moments $\widehat{\mu}(\xi)$ characterise uniquely the measure μ.

A straightforward consequence of self-duality :

Proposition

A probability measure μ is invariant and tempered for the SIP if and only if $\widehat{\mu}$ is bounded harmonic for the SIP, that is,

$$\mathbb{E}_\xi \widehat{\mu}(\xi) = \widehat{\mu}(\xi).$$

Therefore,
study of the invariant measures = identify the set of bounded harmonic functions for the SIP.

To this purpose, we prove a **successful coupling** for the finite SIP. Note that the SIP is not monotone...

Successful coupling

initial locations:

$$\mathbf{x} = (x_1, ..., x_n),\ \mathbf{y} = (y_1, .., y_n).$$

sets of SIP-particles:

$$\mathbf{X}^S(t) = (X_1^S, ..., X_n^S)(t),\ \mathbf{Y}^S(t) = (Y_1^S, ..., Y_n^S)(t).$$

sets of IRW-particles:

$$\mathbf{X}^I(t) = (X_1^I, ..., X_n^I)(t),\ \mathbf{Y}^I(t) = (Y_1^I, ..., Y_n^I)(t).$$

Theorem

There exists a successful coupling for the SIP(m). That is,

$$\widehat{\mathbb{P}}^{SIP}_{\mathbf{x},\mathbf{y}}(\tau < \infty) = 1$$

where $\tau = \inf(t : \mathbf{X}^S(s) = \mathbf{Y}^S(s),$ *for all* $s \geq t)$ *and* $\widehat{\mathbb{P}}^{SIP}_{\mathbf{x},\mathbf{y}}$ *denotes the joint distribution of* $\{(\mathbf{X}^S(t), \mathbf{Y}^S(t)), t \geq 0\}$ *starting from* $(\mathbf{x}, \mathbf{y})$*.*

Collision at time $t > 0$

Within a same set, particles are at neighbouring positions at time t:

$$\{\exists i : 1 \leq i \neq j \leq n, |X_i(t) - X_j(t)| = 1\}$$

Sketch of the proof

1. **Case** $d \geq 3$: any continuous-time simple random walk is transient.
Let $Z(t)$ be a simple random walk with probability transition $p(.,.)$,

$$H(z) := \mathbb{P}_z\big(|Z(t)| > 1,\ \forall t > 0\big) > 0,$$

and $H(z) \to 1$, as t goes to infinity.

Having a collision ?

$$\mathbb{P}_{\mathbf{x}}^{SIP}\Big(\bigcup_{1\leq i\neq j\leq n} \big\{|(X_i - X_j)(t)| = 1\big\}\Big)$$
$$\leq \sum_{1\leq i\neq j\leq n} \mathbb{P}_{|x_i - x_j|}\Big(\{|(X_i - X_j)(t)| = 1\Big)$$

which is small as soon as $|x_i - x_j| > R$, R large enough.
With probability $p(R)$ (such that $p(R) \to 1$, as $R \to \infty$), there is no collision.

Couple the two sets of IRW-particles, via the coordinate-wise Ornstein coupling.

Opoku, A. and Redig, F.: JSP '15
Couple each set of IRW-particles with a set of SIP-particles (each particle having then a corresponding *partner* in the IRW-set): for all $1 \leq i \leq n$, in probability

$$\lim_{t \to \infty} \frac{|X_i^S(t) - X_i^I(t)|^2}{t} = 0$$

So far, the successful coupling occurs with positive probability.

After some time T, start the above coupling, so that any pair of particles within any set is $\alpha(T)$ apart (such that $\alpha(T) \to \infty$, as $T \to \infty$).
Therefore: one has a failed coupling with probability at most

$$\big(1 - p(\alpha(T))\big)\big(1 - \pi(T)\big),$$

which goes to zero as T goes to infinity.

2. **Case** $d = 1, 2$: any continuous-time simple random walk is recurrent. Following the idea of de Masi, A. and Presutti, E. AIHP '83.

Fix $\delta \in (0, 1)$.

- **Step 1:** time window $[0, t - \delta t]$.
  Both set of IRW-particles do the same jumps.
  SIP-particles follow from Opoku-Redig coupling.
  There, w.h.p. any pair of particles within a same set is $O(\sqrt{t})$ apart BUT any pair of IRW-particle and its SIP-particle partner is $o(\sqrt{t})$ apart.

- **Step 2:** time window $[t - \delta t, t]$.
  Proceed a coordinate-wise Ornstein coupling for the IRW- and SIP- particles.
  If no collision occurs: all good.
  If a collision occurs: failed attempt.

Step 1: time window $[0, t - \delta t]$.

- Both set of IRW-particles do the same jumps:

$$\sum_{i=1}^{n} |X_i^I(s) - Y_i^I(s)| = \sum_{i=1}^{n} |x_i - y_i| = k_n, \quad \text{for all s}$$

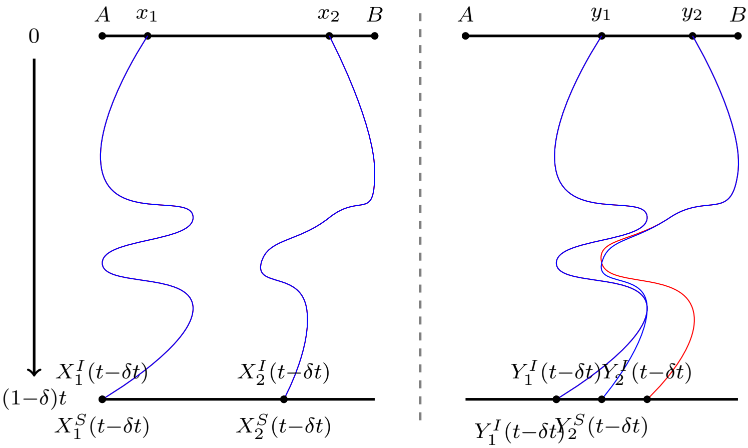
- By Opoku-Redig '15 coupling: with probability close to one,

$$\sum_{i=1}^{n} |X_i^S(s) - X_i^I(s)| + |Y_i^S(s) - Y_i^I(s)| \leq \psi(s)$$

where $\psi(s)/\sqrt{s} \to 0$ as $s \to \infty$.

Thus, with probability close to one,

$$\begin{aligned}
&\sum_{i=1}^{n} |X_i^S(s) - Y_i^S(s)| \\
&\quad \leq \sum_{i=1}^{n} |X_i^S(s) - X_i^I(s)| + |X_i^I(s) - Y_i^I(s)| + |Y_i^I(s) - Y_i^S(s)| \\
&\quad \leq k_n + 2\psi(s).
\end{aligned}$$



SIP-particles $\mathbf{X}^S$ and $\mathbf{Y}^S$, IRW-particles $\mathbf{X}^I$ and $\mathbf{Y}^I$.
Each SIP-particle is distant from an IRW-partner by $\mathbf{o}(\sqrt{\mathbf{t}})$ and coupled thanks to Opoku&Redig's coupling.
If no collision occurs, IRW- and SIP- particles follow the same path.
At time $(1-\delta)t$, the distance between any pair of IRW-particle is $\mathbf{O}(\sqrt{\mathbf{t}})$.

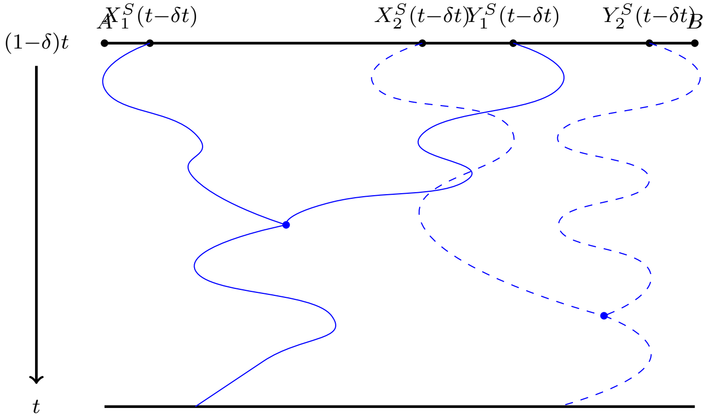
Step 2: time window $[t-\delta t, t]$.
At $(t-\delta t)$, any pair of SIP- as well as IRW- particles are $\sqrt{t}$ apart with probability close to one, t large enough.

It is then enough to show: a collision (in any set of SIP-particles) occurs before X^S and Y^S are coupled is very unlikely.

Notice $\{|X_i^I(s) - Y_i^I(s)|, s \geq 0\}$ acts as a random walk following $p(\cdot,\cdot)$ and moving at rate m, for all $1 \leq i \leq n$.
Let τ_0^1 (resp. τ_1^2) stand for the hitting time of 0 (resp. of 1) for such an independent random walker. And let us prove

$$\lim_{t\to\infty} \widehat{\mathbb{P}}^{IRW}_{\psi(t),\sqrt{t}}\big(\tau_0^1 \geq \tau_1^2\big) = 0,$$

where $\widehat{\mathbb{P}}^{IRW}_{a,b}$ denotes the joint distribution of two random walkers starting respectively from a and b.



While the IRW-particles $\mathbf{X}^I$ and $\mathbf{Y}^I$ are coupled (there, omitted), the SIP-particles are coupled in the same way **provided** no collision occurs within any set of particles (that is, here, between X_1^S and X_2^S or Y_1^S and Y_2^S, but any other does not matter).

Choose a time scale $\phi(t)$ such that $\phi(t) \to \infty$ as $t \to \infty$, and write

$$\begin{aligned}
&\widehat{\mathbb{P}}^{IRW}_{\psi(t),\sqrt{t}}\big(\tau_0^1 \geq \tau_0^2\big) \\
&\qquad \leq \widehat{\mathbb{P}}^{IRW}_{\psi(t),\sqrt{t}}\Big(\tau_0^1 \geq \tau_0^2, \tau_0^2 \geq \phi(t)\Big) + \widehat{\mathbb{P}}^{IRW}_{\psi(t),\sqrt{t}}\Big(\tau_0^1 \geq \tau_0^2, \tau_0^2 \leq \phi(t)\Big)
\end{aligned}$$

Therefore, it is enough to estimate:

$$\mathbb{P}^{IRW}_{\psi(t)}\Big(\tau_0^1 \geq \phi(t)\Big) + \mathbb{P}^{IRW}_{\sqrt{t}}\Big(\tau_0^2 \leq \phi(t)\Big)$$

By reflection principle, it is equal to

$$\mathbb{P}^{IRW}_{0}\Big(|Z(\phi(t))| \leq \psi(t)\Big) + \mathbb{P}^{IRW}_{0}\Big(|Z(\phi(t))| \geq \sqrt{t}\Big)$$

Equivalently,

$$\mathbb{P}_0^{IRW}\left(\frac{|Z(\phi(t))|}{\psi(t)} \leq 1\right) + \mathbb{P}_0^{IRW}\left(\frac{|Z(\phi(t))|}{\sqrt{t}} \geq 1\right)$$

which goes to zero as $t \to \infty$:
choose $\phi(t) = c\psi(t)^2$, the first term vanishes by invariance principle and letting c go to infinity while the second term vanishes since $\psi(t) = o(\sqrt{t})$.

We proved a successful coupling of two finite SIP is positive in time lapse $[t - \delta t, t]$. Now, iterate independently such coupling attempts.

$\square$

Corollary

Let μ be a tempered invariant measure for the SIP. Then, $\widehat{\mu}(\xi)$ is constant, i.e. depends only on $|\xi| = n$.

Proof.

Consider two sets of SIP-particles $\mathbf{X}^S$ and $\mathbf{Y}^S$ starting resp. from $\mathbf{x}$ and $\mathbf{y}$.

$$\begin{aligned}\widehat{\mu}(\mathbf{x}) &= \mathbb{E}_{\mathbf{x}}^{SIP}\widehat{\mu}(\mathbf{X}^S(t)) \\ &= \widehat{\mathbb{E}}_{\mathbf{x},\mathbf{y}}^{SIP}\Big(\widehat{\mu}(\mathbf{Y}^S(t))\mathbf{1}\{\mathbf{X}^S(t) = \mathbf{Y}^S(t)\}\Big) \\ &\quad + \widehat{\mathbb{E}}_{\mathbf{x},\mathbf{y}}^{SIP}\Big(\widehat{\mu}(\mathbf{X}^S(t))\mathbf{1}\{\mathbf{X}^S(t) \neq \mathbf{Y}^S(t)\}\Big)\end{aligned}$$

After some computations, we get

$$|\widehat{\mu}(\mathbf{x}) - \widehat{\mu}(\mathbf{y})| \leq cst \cdot \widehat{\mathbb{P}}_{x,y}^{SIP}\big(\mathbf{X}^S(t) \neq \mathbf{Y}^S(t)\big).$$

$\square$

Characterization of the invariant measures within a certain class

Theorem

If μ is an invariant tempered ergodic probability measure, then $\mu = \nu_\lambda^m$. As a consequence,

$$\left(\mathfrak{I}_{temp}\right)_e = \left\{\nu_\lambda^m, \lambda \in [0,1)\right\}.$$

SKETCH OF THE PROOF

Lemma

The ergodic elements of $\mathfrak{I}_{temp}$ are exactly the extreme points of $\mathfrak{I}_{temp}$,

By the successful coupling: if μ is invariant tempered, then $\widehat{\mu}(\xi)$ depends only on $|\xi|$.

Recall the measure μ is uniquely determined by its D-transform. It is enough to show

$$\widehat{\mu}(n) = \alpha^n \text{ for some } \alpha \geq 0.$$

This actually follows from the following argument

$$\widehat{\mu}(a+b) = \widehat{\mu}(a)\widehat{\mu}(b), \quad \text{for all } a, b \in \mathbb{N}.$$

$\square$

A few remarks

Above results are extendible to related processes dual to the SIP, such as the Brownian Energy Process: its tempered invariant ergodic measures are product of Gamma distributions.
See Giardinà, C. ; Kurchan, J. ; Redig, F. and Vafayi, K.: JSP '09.

Other processes related to the SIP concern the *thermalised SIP* and the *thermalised BEP* (generalised KMP process, Kipnis, C. ; Marchioro, C. and Presutti, E. JSP '82).
Here, several particles can jump at the same time (instantaneous redistribution). If the particles are sufficiently spread out, the above arguments still hold.
See Carinci, G. ; Giardinà, C. ; Giberti, C. and Redig, F.: JSP '13.

Thanks for your attention

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