Percolation games, probabilistic cellular automata, and the hard-core model

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Nancy, France

Joint work with James B. Martin and Alexander E. Holroyd

Rencontres de Probabilités, Rouen, September 17, 2015

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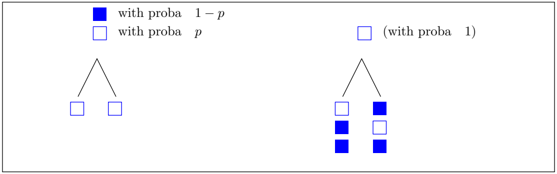

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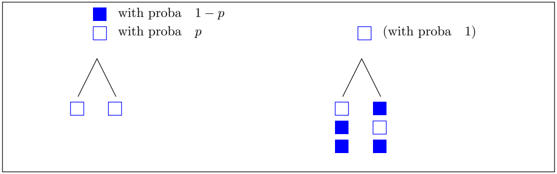

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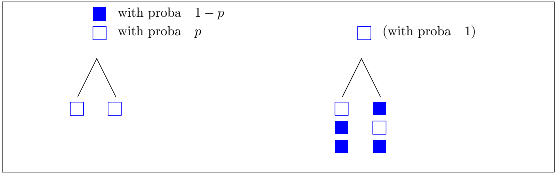

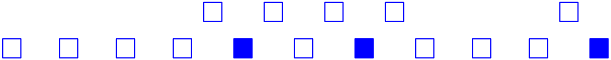

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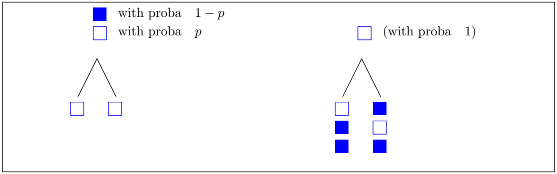

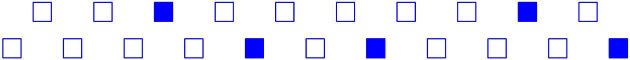

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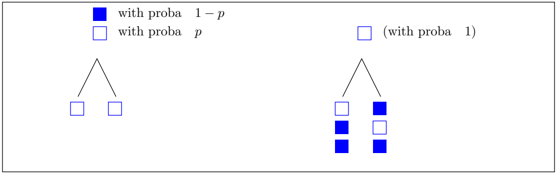

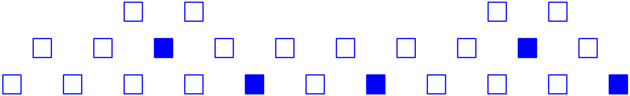

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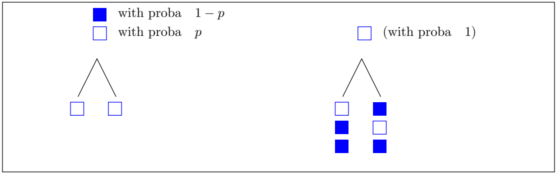

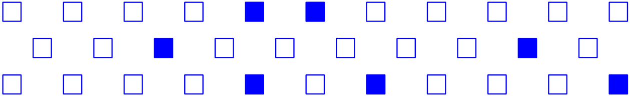

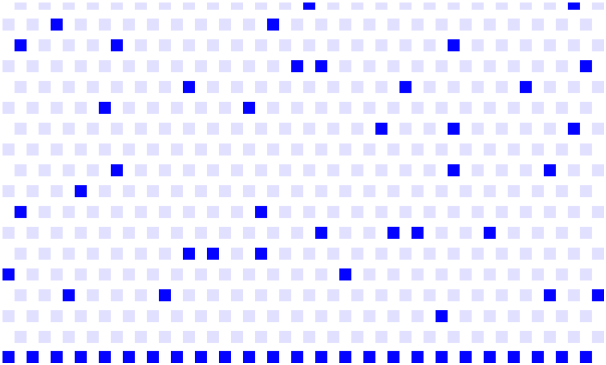

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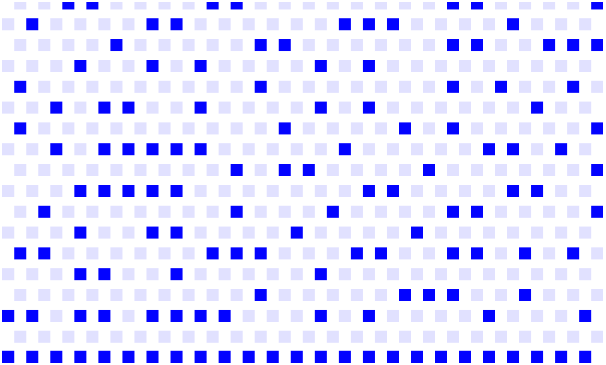

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If the system **forgets** its initial configuration, we say it is ergodic.

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The PCA F on $\{0,1\}^{\mathbb{Z}}$ is ergodic if:

- it has a **unique invariant measure** $\pi \in \mathcal{M}(\{0,1\}^{\mathbb{Z}})$, such that $\pi F = \pi$,
- for any initial measure $\mu \in \mathcal{M}(\{0,1\}^{\mathbb{Z}})$, the sequence of iterates $(\mu F^n)_{n \geq 0}$ **converges** weakly to π.

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For which values of the parameter p is the PCA ergodic?
How can we describe its invariant measure(s)?

Motivations

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- A model very easy to define!

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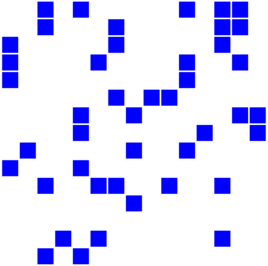
Outline

Definition of the percolation game

Grid $\mathbb{N} \times \mathbb{N}$, with each site colored in blue independently with probability p (here, $p = 0.2$).



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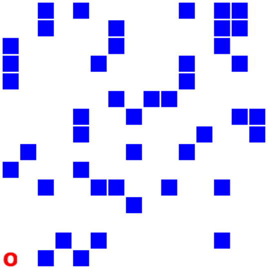

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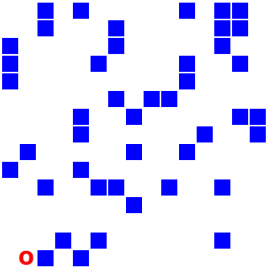

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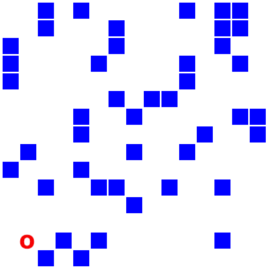

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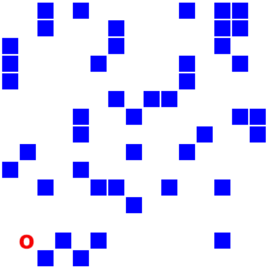

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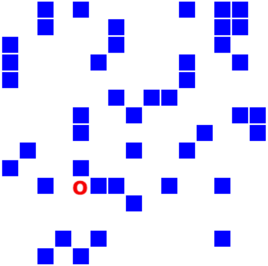

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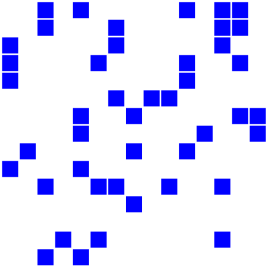
Winning positions, loss and draws

A position is:

- a win (**W**) if from this position, the player whose turn it is to play has a winning strategy,
- a loss (**L**) if from this position, the other player has a winning strategy,
- a draw (**D**) if neither player has a winning strategy, so that with "best play", the game will continue for ever.



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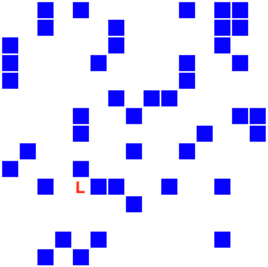

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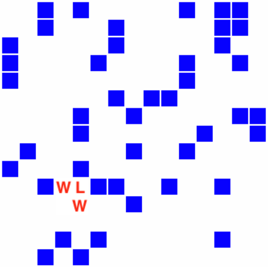

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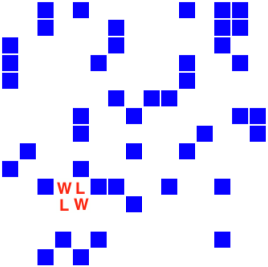

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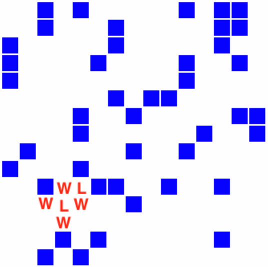

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Are there values of $p > 0$ for which there are **D** with a positive probability?
What is the probability for the origin to be **W**, **L**, or **D**?

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If there are no **D**, the PCA we obtain is defined as follows.

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The **D** play the role of symbols "?".

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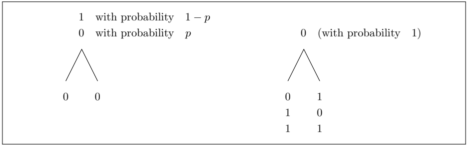

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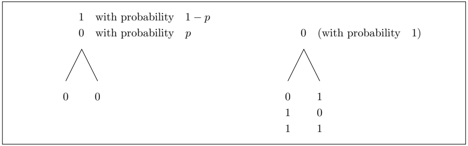

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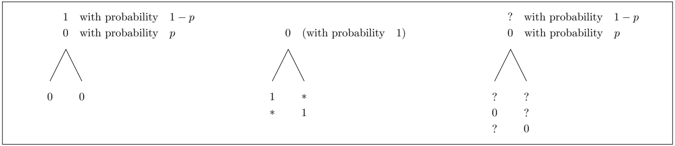

ACP F_p

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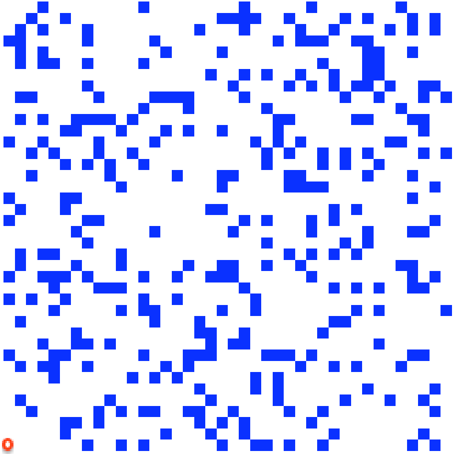

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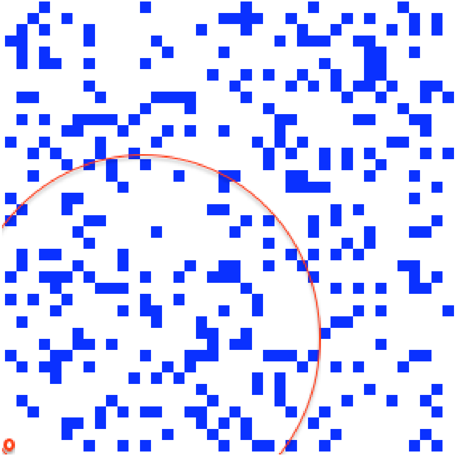

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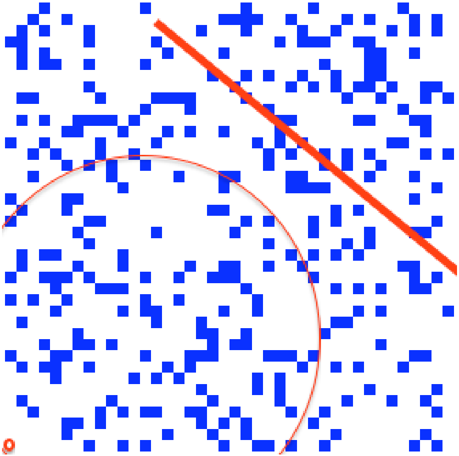

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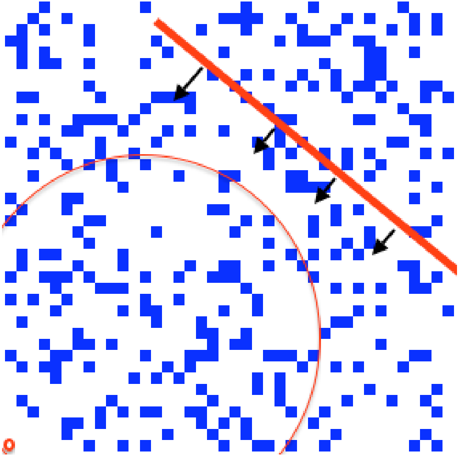

Outline

Markovian invariant measure

One can show that for any value of p, the PCA has a Markovian invariant measure μ_p, given by the following transition matrix.

$$P = \begin{pmatrix} p_{0,0} & p_{0,1} \\ p_{1,0} & p_{1,1} \end{pmatrix} = \begin{pmatrix} \frac{2-p-\sqrt{p(4-3p)}}{2(1-p)^2} & \frac{2p^2-3p+\sqrt{p(4-3p)}}{2(1-p)^2} \\ \frac{-p+\sqrt{p(4-3p)}}{2(1-p)} & \frac{2-p-\sqrt{p(4-3p)}}{2(1-p)} \end{pmatrix}$$

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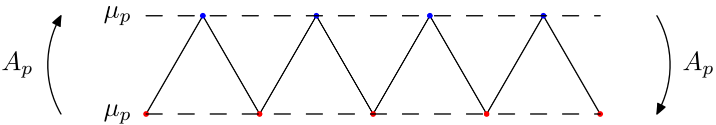

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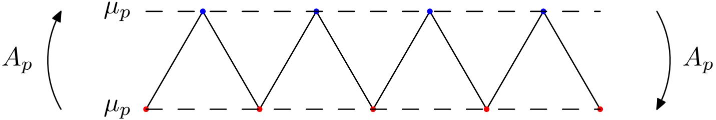
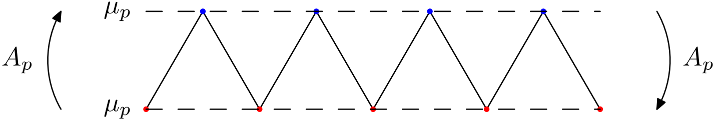


Markovian invariant measure

Reversible invariant measures of the PCA are Gibbs measures on the **doubling graph** (="accordion" graph) such that:

- there are no two consecutive 1's
- the probability to have a 1 if the two neighbours are in state 0 is equal to $1 - p$.



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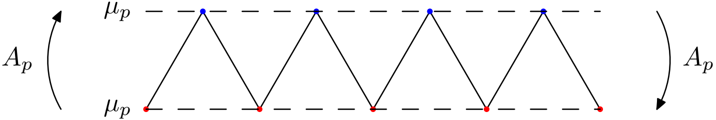
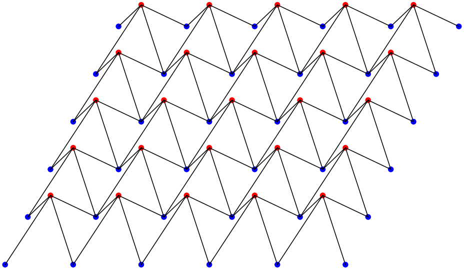


If we unfold the accordion graph, we recover the Gibbs measures of the one-dimensional hardcore model!

Dimension 2



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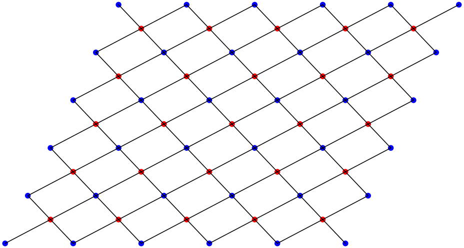

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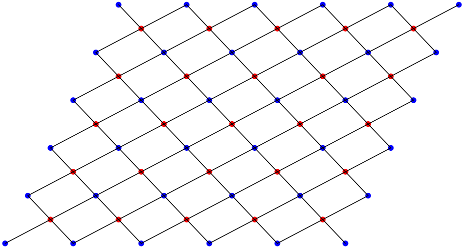

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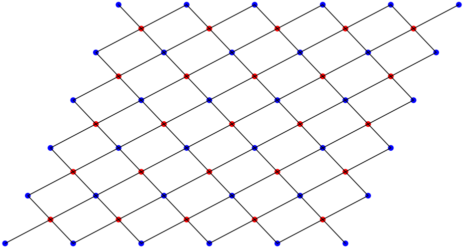

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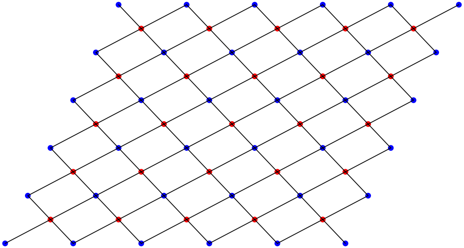

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Multiplicity of Gibbs measures.

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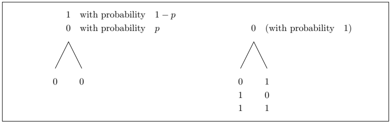
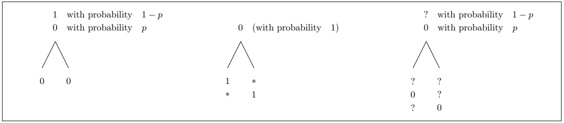
Theorem

For any $p \in (0, 1)$, the PCA A_p is ergodic and the probability of draws is 0 for the percolation game on $\mathbb{N}^2$.

We show that for any p, the PCA A_p is ergodic.



PCA A_p



PCA F_p

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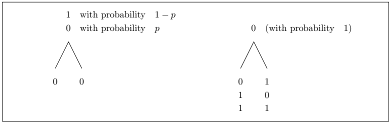

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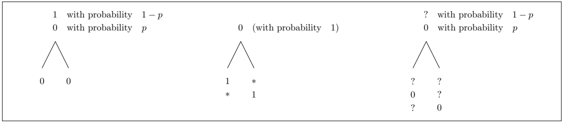

PCA F_p

Step 1

It is enough to show that starting from the configuration with only "?", when iterating F_p, the density of "?" converges to 0 (coupling of all the trajectories for A_p).

Step 2

We show that F_p has no invariant measure for which there is a positive density of symbols “?”.

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Step 3

Let μ be an invariant measure of F_p.
We introduce a weight on symbols "?" under μ, the weighting of each "?" depending on its neigbours.
We show that this quantity decreases under F_p.

Weighting system

Right-weight of a symbol “?” =

- 3 if it is followed by a 0, then by a 1,
- 2 if it is followed by a 0, then by something else than a 1,
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Total weight =left-weight + right-weight.

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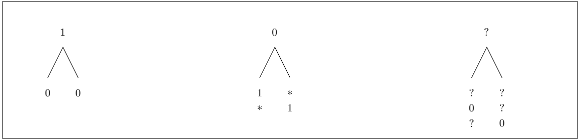
Example: in 10??10, the first “?” has a weight 3+1=4 and the second one a weight $1 + 1 = 2$.

$F_p = R_p \circ D$



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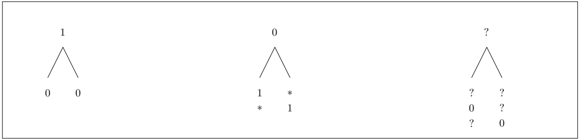

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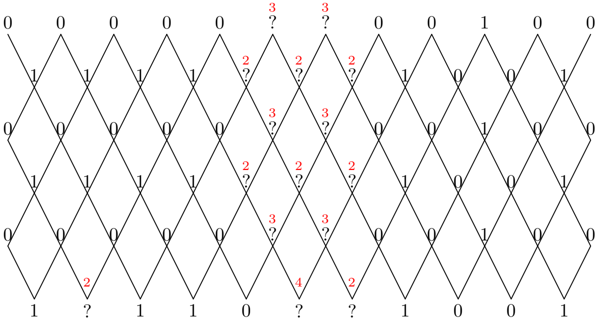
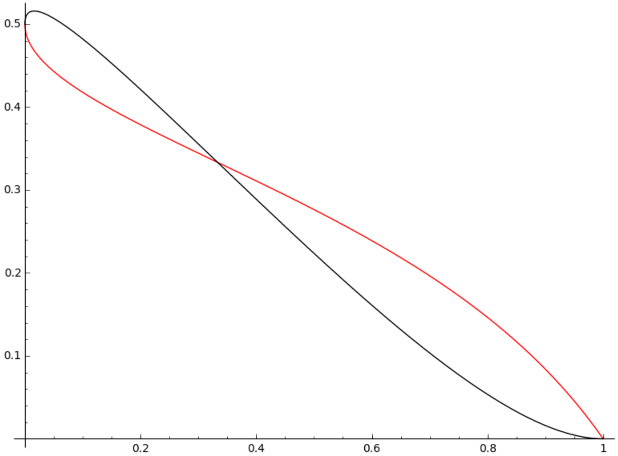
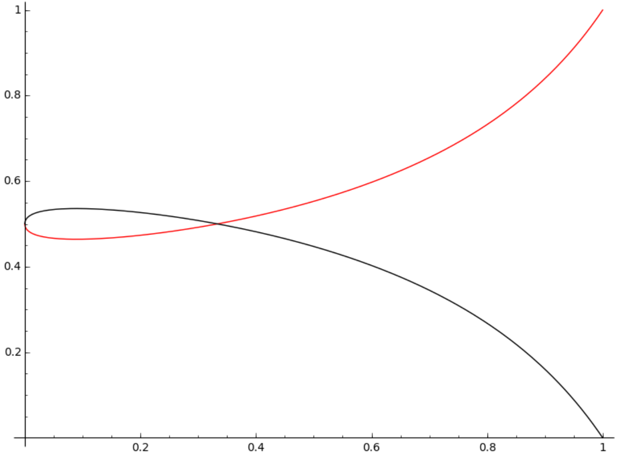




Red: winning probability ; black: loss probability .



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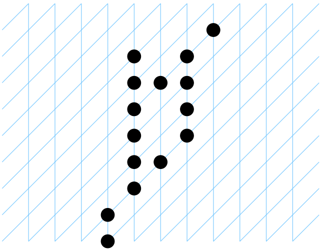
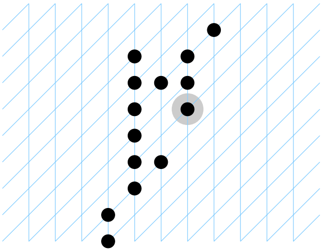
- How can we describe its invariant measure(s)?
- In dimension 1, for **elementary PCA** (neighbourhood of size 2, binary states), is it true that if all the probability transitions are in $(0, 1)$, then the PCA is ergodic?

Definition of directed animals

Directed animal of **base** C: finite subset of vertices of $\mathbb{Z} \times \mathbb{N}$, connected from $C \times \{0\}$ by links $\uparrow$ or $\nearrow$



A directed animal
(whose base has only one element)



Not a directed animal

Enumeration of directed animals

Counting series of directed animals of base C:

$$S_C(x) = \sum_{E\text{:DA dof base } C} x^{|E|} = \sum_{n \geq 0} a_n(C) x^n,$$

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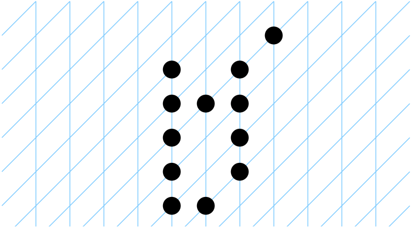
Recurrence relation: $S_C(x) = x^{|C|} \left(\sum_{D \subset C + \{0,1\}} S_D(x) \right)$



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Let μ be an invariant measure of the PCA of parameter p, and let $X, Y \sim \mu$.

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References: D. Dhar, M. Bousquet-Mélou, J.-F. Marckert, Y. Le Borgne, M. Albenque...