From the quantum Lie algebra $U_q(\mathfrak{sl}_2)$ to the ASEP(q, j)

Gioia Carinci

joint work with: Cristian Giardinà (Modena), Frank Redig (Delft), Tomohiro Sasamoto (Tokyo)

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Outline

- Constructive approach to duality theory via Lie algebra
- $\mathfrak{su}_q(2)$ algebra: construction of ASEP(q, j)
- Properties of ASEP(q, j)

1. Constructive approach

to duality theory

via Lie algebra

Stochastic Duality

$(\eta_t)_{t\geq 0}$ Markov process on Ω with generator L,

$(\xi_t)_{t\geq 0}$ Markov process on Ω_{dual} with generator L_{dual}

ξ_t is dual to η_t with duality function $D : \Omega \times \Omega_{dual} \to \mathbb{R}$ if $\forall t \geq 0$

$$\mathbb{E}_\eta(D(\eta_t, \xi)) = \mathbb{E}_\xi(D(\eta, \xi_t)) \qquad \forall (\eta, \xi) \in \Omega \times \Omega_{dual}$$

η_t is self-dual if $L_{dual} = L$.

Duality is equivalent to $\quad LD(\cdot, \xi)(\eta) = L_{dual}D(\eta, \cdot)(\xi)$

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- Symmetries and (non-trivial) self-duality:

  S: symmetry of the Markov generator, i.e. $[\mathbf{L}, \mathbf{S}] = 0$

  $\mathbf{d}$: trivial self-duality function

  $\longrightarrow$ $\mathbf{D} = \mathbf{Sd}$ is a self-duality function

Lie algebra

A Lie algebra is a vector space $\mathfrak{g}$ over a field F with a binary operation $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ (Lie bracket)

- $[\cdot,\cdot]$ is bilinear
- $\forall\ u, v$ in $\mathfrak{g}$: $[u, v] = -[v, u]$
- [Jacobi identity]: $\forall\ u, v, w$ in $\mathfrak{g}$

$$[u, [v, w]] + [w, [u, v]] + [v, [w, u]] = 0$$

Algebraic approach

1. Write the Markov generator in abstract form, i.e. as an element of a Lie algebra, using the algebra generators.

2. Duality is related to a change of representation.
   Duality functions are the intertwiners.

3. Self-duality is associated to symmetries.

Conversely, the approach can be turned into a constructive method.

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vi) (*Markov generator*): Apply a "ground state transformation" to turn H into a Markov generator L.

2. $\mathfrak{su}_q(2)$ algebra:

construction of ASEP(q, j)

q-numbers

For $q \in (0,1)$ and $n \in \mathbb{N}_0$ introduce the q-number

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$$

Remark: $\lim_{q\to 1}[n]_q = n$.

The first q-number's are:

$$[0]_q = 0, \qquad [1]_q = 1, \qquad [2]_q = q+q^{-1}, \qquad [3]_q = q^2+1+q^{-2}, \quad \ldots$$

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For $q \in (0, 1)$ consider the algebra with generators J^+, J^-, J^0

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A standard representation $(n = 0, 1, \dots 2j)$

$$\begin{cases} J^+ \,|n\rangle &= \sqrt{[2j-n]_q [n+1]_q}\,|n+1\rangle \\ J^- \,|n\rangle &= \sqrt{[n]_q [2j-n+1]_q}\,|n-1\rangle \\ J^0 \,|n\rangle &= (n-j)\,|n\rangle \end{cases}$$

In this representation $C\,|n\rangle = [j]_q [j+1]_q \,|n\rangle$

Co-product

A coproduct on an algebra $\mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ is an algebra homomorphism:

$$\Delta([A, B]) = [\Delta(A), \Delta(B)] \qquad \forall A, B \in \mathfrak{g}$$

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Define the co-product $\Delta : U_q(\mathfrak{sl}_2) \to U_q(\mathfrak{sl}_2)^{\otimes 2}$ as follows

$$\begin{aligned} \Delta(J^\pm) &= J^\pm \otimes q^{-J^0} + q^{J^0} \otimes J^\pm \\ \Delta(J^0) &= J^0 \otimes 1 + 1 \otimes J^0 \end{aligned}$$

then

$$[\Delta(J^+), \Delta(J^-)] = [2\Delta(J^0)]_q \qquad [\Delta(J^0), \Delta(J^\pm)] = \pm\Delta(J^\pm)$$

Quantum Hamiltonian

$$\Delta(C_i) = -q^{J_i^0}\left\{J_i^+ \otimes J_{i+1}^- + J_i^- \otimes J_{i+1}^+ + B_{i,i+1}\right\}q^{-J_{i+1}^0}$$

$$\begin{aligned} B_{i,i+1} &= \frac{(q^j + q^{-j})(q^{j+1} + q^{-(j+1)})}{2(q-q^{-1})^2}\left(q^{J_i^0} - q^{-J_i^0}\right) \otimes \left(q^{J_{i+1}^0} - q^{-J_{i+1}^0}\right) \\ &+ \frac{(q^j - q^{-j})(q^{j+1} - q^{-(j+1)})}{2(q-q^{-1})^2}\left(q^{J_i^0} + q^{-J_i^0}\right) \otimes \left(q^{J_{i+1}^0} + q^{-J_{i+1}^0}\right) \end{aligned}$$

$$H_{(L)} := \sum_{i=1}^{L-1}\left(1^{\otimes(i-1)} \otimes \Delta(C_i) \otimes 1^{\otimes(L-i-1)} + c_{q,j}1^{\otimes L}\right)$$

$$c_{q,j} = \frac{(q^{2j} - q^{-2j})(q^{2j+1} - q^{-(2j+1)})}{(q-q^{-1})^2} \qquad s.t. \qquad H\left(\otimes_{i=1}^{L}|0\rangle\right) = 0$$

Symmetries of H

Iterate the co-product $\Delta^n : U_q(\mathfrak{sl}_2) \to U_q(\mathfrak{sl}_2)^{\otimes(n+1)}$, i.e. for $n \geq 2$

$$\begin{aligned}
\Delta^n(J^\pm) &= \Delta^{n-1}(J^\pm) \otimes q^{-J^0} + q^{\Delta^{n-1}(J^0)} \otimes J^\pm \\
\Delta^n(J^0) &= \Delta^{n-1}(J^0) \otimes 1 + 1^{\otimes n} \otimes J^0
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Lemma

$$\begin{aligned} \mathcal{J}^\pm &:= \Delta^{L-1}(J^\pm) = \sum_{i=1}^{L} q^{J_1^0} \otimes \cdots \otimes q^{J_{i-1}^0} \otimes J_i^\pm \otimes q^{-J_{i+1}^0} \otimes \ldots \otimes q^{-J_L^0} \\ \mathcal{J}^0 &:= \Delta^{L-1}(J^0) = \sum_{i=1}^{L} \underbrace{1 \otimes \cdots \otimes 1}_{(i-1)\text{ times}} \otimes J_i^0 \otimes \underbrace{1 \otimes \cdots \otimes 1}_{(L-i)\text{ times}} \,. \end{aligned}$$

are symmetries of $H_{(L)}$.

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- construct a non-trivial symmetry of L using the fact that if S is a symmetry of H then $G^{-1} S G$ is a symmetry of L
- use the non-trivial symmetries of L to construct self duality functions for the associated Markov process

Ground State transformation

Lemma

Let H be a matrix with $H(\eta,\eta') \geq 0$ for $\eta \neq \eta'$.
Suppose g is a positive ground state. i.e. $H\,g = 0$ and $g(\eta) > 0$.
Let G be the matrix $G(\eta,\eta') = g(\eta)\delta(\eta,\eta')$. Then

$$L = G^{-1} H\,G$$

is a Markov generator.

Proof.

$$L(\eta,\eta') = \frac{H(\eta,\eta')g(\eta')}{g(\eta)}$$

Therefore

$$L(\eta,\eta') \geq 0 \quad \text{if} \quad \eta \neq \eta' \qquad\qquad \sum_{\eta'} L(\eta,\eta') = 0$$

Exponential symmetries

- $g^{(0)} = \otimes_{i=1}^{L} |0\rangle$ is a ground state, i.e. $Hg^{(0)} = 0$.
- For every symmetry $[H, S] = 0$ another ground state is $g = Sg^{(0)}$.
- The exponential symmetry

$$S^{+} = \exp_{q^2}(E) = \sum_{n \geq 0} \frac{(E)^n}{[n]_q!} q^{-n(n-1)/2}$$

where

$$E = \Delta^{(L-1)}(q^{J^0} J^{+})$$

gives a positive ground state

$$g = S^{+} g^{(0)} = \sum_{\ell_1, \ldots, \ell_L} \otimes_{i=1}^{L} \left(\sqrt{\binom{2j}{\ell_i}_q} \cdot q^{\ell_i(1+j-2ji)} \right) |\ell_i\rangle$$

ASEP(q,j) process

Definition

The Markov process ASEP(q,j) on $[1,L]\cap\mathbb{Z}$, denoted by $(\eta(t))_{t\geq 0}$, with state space $\{0,1,\ldots,2j\}^L$ is defined by

$$(L^{ASEP(q,j)}f)(\eta)=\sum_{i=1}^{L-1}(L_{i,i+1}f)(\eta)$$

with

$$\begin{aligned}(L_{i,i+1}f)(\eta) &= q^{\eta_i-\eta_{i+1}-(2j+1)}[\eta_i]_q[2j-\eta_{i+1}]_q(f(\eta^{i,i+1})-f(\eta)) \\ &+ q^{\eta_i-\eta_{i+1}+(2j+1)}[2j-\eta_i]_q[\eta_{i+1}]_q(f(\eta^{i+1,i})-f(\eta))\end{aligned}$$

Remark: it follows from $L=G^{-1}HG$.

ASEP(q,j) process: special cases

$$\begin{aligned}(L_{i,i+1}f)(\eta) &= q^{\eta_i-\eta_{i+1}-(2j+1)}[\eta_i]_q[2j-\eta_{i+1}]_q(f(\eta^{i,i+1})-f(\eta)) \\ &+ q^{\eta_i-\eta_{i+1}+(2j+1)}[2j-\eta_i]_q[\eta_{i+1}]_q(f(\eta^{i+1,i})-f(\eta))\end{aligned}$$

- $q = 1 \rightarrow SEP(j)$: symmetric partial exclusion
  jump right at rate $\eta_i(2j-\eta_{i+1})$, jump left at rate $(2j-\eta_i)\eta_{i+1}$
- $j = 1/2 \rightarrow ASEP(q)$: asymmetric exclusion
  jump right at rate q^{-1} , jump left at rate q
- $j = \infty \rightarrow TAZRP(q)$: totally asymmetric zero range
  after rescaling time $t \rightarrow q^{4j-1}t$, jump right at rate $\frac{1-q^{2\eta_i}}{1-q^2}$

3. Properties of ASEP(q, j)

Properties of ASEP(q,j)

Theorem

a) The $ASEP(q,j)$ is well-defined on $\mathbb{Z}$ and is a monotone process.

b) The $ASEP(q,j)$ on $\mathbb{Z}$ has a family (labeled by $\alpha > 0$) of reversible product measures with marginals

$$\mathbb{P}_\alpha(\eta_i = x) = \frac{\alpha^x}{Z_{i,\alpha}} \binom{2j}{x}_q \cdot q^{2x(1+j-2ji)}$$

c) The $ASEP(q,j)$ has translation invariant stationary product measures only for $j = 1/2$ and for $j \to \infty$.

Proof of b) for $\alpha = 1$: using $H = H^T$

$$G^2 L = (G^2 G^{-1} H G) = (G H G^{-1}) G^2 = L^T G^2$$

Self-duality of ASEP(q, j)

Theorem

The ASEP(q, j) on $\mathbb{Z}$ is self-dual on

$$D(\eta, \xi) = \prod_{i=1}^{L} \frac{[\eta_i]_q!}{[\eta_i - \xi_i]_q!} \frac{\Gamma_q(2j + 1 - \xi_i)}{\Gamma_q(2j + 1)} \cdot q^{(\eta_i - \xi_i)[2\sum_{k=1}^{i-1} \xi_k + \xi_i] + 4ji\xi_i} \cdot 1_{\eta_i \geq \xi_i}$$

Proof: it follows from the general method

- $d = G^{-2}$ is a trivial duality function
- $[H, S^+]$ and $L = G^{-1}HG$, thus $[L, G^{-1}S^+G] = 0$.
- $D = (G^{-1}S^+G)G^{-2} = G^{-1}S^+G^{-1}$ is a duality fct.

Current of ASEP(q,j)

Definition

Let
$$N_i(t) := \sum_{k \geq i} \eta_k(t)$$
The current $J_i(t)$ during the time interval $[0,t]$ across the bond $(i-1,i)$ is defined as the net number of particles traversing the bond in the right direction:
$$J_i(t) = N_i(t) - N_i(0)$$

Remark: let $\xi^{(i)}$ be the configuration with 1 dual particle:
$$\xi_m^{(i)} = \begin{cases} 1 & \text{if } m = i \\ 0 & \text{otherwise} \end{cases}$$
then
$$D(\eta, \xi^{(i)}) = \frac{q^{4ji-1}}{q^{2j} - q^{-2j}} \cdot \left(q^{2N_i} - q^{2N_{i+1}}\right)$$

First q^2-moment of the current

Theorem

$$\mathbb{E}_\eta\left[q^{2J_i(t)}\right] = q^{\sum_{k<i}\eta_k} - \sum_{k<i} q^{-4jk}\,\mathbb{E}\left[q^{4jX(t)}\left(1-q^{-2\eta_{X(t)}}\right) q^{2(N_{X(t)}(0)-N_i(0))} \mid X(0)=k\right]$$

with $X(t)$ a random walker on $\mathbb{Z}$ jumping left at rate $q^{2j}[2j]_q$ and jumping right at rate $q^{-2j}[2j]_q$

$$\mathbb{P}(X(t)=z \mid X(0)=k) = e^{-[4j]_q t} q^{-2j(z-k)} I_{z-i}(2[2j]_q t)$$

$I_n(t)$ modified Bessel fct.

First q^2-moment of the current

Proof: Duaility gives

$$\mathbb{E}_\eta(D(\eta(t),\xi^{(i)})) = \mathbb{E}_{\xi^{(i)}}(D(\eta,\xi^{(X(t))})$$

$$\mathbb{E}_\eta\left[q^{4ji}\cdot(q^{2N_i(t)}-q^{2N_{i+1}(t)})\right] = \mathbb{E}_{\xi^{(i)}}\left[q^{4jX(t)}\cdot(q^{2N_{X(t)}}-q^{2N_{X(t)+1}})\right]$$

Therefore

$$\mathbb{E}_\eta[q^{2N_i(t)}] = \mathbb{E}_\eta[q^{2N_{i+1}(t)}] + q^{-4ji}\mathbb{E}_{\xi^{(i)}}\left[q^{4jX(t)}\cdot(q^{2N_{X(t)}}-q^{2N_{X(t)+1}})\right]$$

Multiply by $q^{-2N_i(0)}$ to get a recursion relation for the current and iterate.

Step initial condition

Proposition

For the step initial conditions $\eta^{\pm} \in \{0, \ldots, 2j\}^{\mathbb{Z}}$ defined as

$$\eta_i^+ := \begin{cases} 0 & \text{for } i < 0 \\ 2j & \text{for } i \geq 0 \end{cases} \qquad \eta_i^- := \begin{cases} 2j & \text{for } i < 0 \\ 0 & \text{for } i \geq 0 \end{cases}$$

one has

$$\mathbb{E}_{\eta^+}\left[q^{2J_i(t)}\right] = q^{4j\max\{0,i\}}\left\{1 + q^{-4ji}\,\mathbf{E}_i\left[\left(1 - q^{4jX(t)}\right)\mathbf{1}_{X(t)\geq 1}\right]\right\}$$

$$\mathbb{E}_{\eta^-}\left[q^{2J_i(t)}\right] = q^{-4j\max\{0,i\}}\left\{1 - \mathbf{E}_i\left[\left(1 - q^{4jX(t)}\right)\mathbf{1}_{X(t)\geq 1}\right]\right\}$$

Step initial condition

Remark 1: asymptotics

$$\lim_{t\to\infty} \mathbb{E}_{\eta^+}\left[q^{2J_i(t)}\right] = q^{4j\max\{0,i\}}\left(1+q^{-4ji}\right) \qquad \text{shock}$$

$$\lim_{t\to\infty} \mathbb{E}_{\eta^-}\left[q^{2J_i(t)}\right] = 0 \qquad \text{rarefaction fan}$$

Remark 2: contour integral

$$\mathbb{E}_{\eta^+}\left[q^{2J_k(t)}\right] = \frac{q^{4j\max\{0,k\}}}{2\pi i}\int e^{-\frac{q^{2j}[2j]_q^3(q^{-1}-q)^2 z}{(1+q^{4j}z)(1+z)}t}\left(\frac{1+z}{1+q^{4j}z}\right)^k \frac{dz}{z}$$

where the integration contour includes 0 and $-q^{-4j}$ but does not include -1.