Duality and exactly solvable models in non-equilibrium

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September 15, 2015

Abstract

We review recent work on a constructive Lie algebraic approach to duality. The first part is an overview of symmetric models, the second deals with their Lie algebraic structure and the last part deals with the corresponding asymmetric processes obtained via q-deformation of the corresponding Lie algebra. Examples include processes modelling heat conduction, particle transport models of exclusion and inclusion type, models of population dynamics and agent based wealth distribution models.

Contents

1 Introduction

Duality is a technique that has been used in interacting particle systems [21], [15], and stochastic models of population genetics [9]. In [18] and [19] it was realized that dualities in the context of both the symmetric and the asymmetric exclusion process appear to be a consequence of the presence of so-called "non-abelian symmetries", i.e., operators commuting with the generator. This was further exploited for $SU(2)$ with spin 1/2 representations in [16]. In [11, 10] we showed several new dualities, using both $SU(2)$ (exclusion type processes), $SU(1,1)$ (inclusion type processes) and Heisenberg algebra (independent walkers), and established a general relation between the existence of operators that commute with the generator and self-duality functions. This relation makes it possible to constructively search for generators that have a rich set of commuting operators. The set of commuting operators is naturally an algebra, and endowed with the commutator is a Lie algebra. It is therefore natural to think that generators with a rich class of commuting operators can be constructed from Lie algebraic considerations. This is implemented by starting from a central element in a Lie algebra and correctly "lifting" it to act on more variables via a co-product (an operation conserving zero commutators). Natural central elements are given by the Casimir element(s). Going to a concrete representation of the Lie algebra then yields (in many cases) an operator that is already close to a generator. More precisely, it is a generator plus central elements which can be omitted from the point of view of symmetries, or it is a generator plus a multiplication operator ("mass is not conserved") which can be turned into a generator by a so-called ground state transformation (if a strictly positive ground state exists). The second scenario is usually what happens if one passes from the symmetric processes (classical Lie algebras) to their asymmetric counterparts, which amounts to consider the q-deformed algebras. We will illustrate here this technique and its applications in examples from no-equilibrium particle systems and heat conduction models.

2 Lecture 1: Symmetric Models

A stochastic non-equilibrium chain (= one dimensional model) is a model on the state space $K^{\{1,\dots,N\}}$ with generator of the form

$$L = \mathscr{L}_1 + \sum_{i=1}^{N-1} L_{i,i+1} + \mathscr{L}_N \tag{1}$$

here *the bulk part* $\sum_{i=1}^{N-1} L_{i,i+1}$ models transport between lattice sites $i, i+1$, and the *boundary part* $\mathscr{L}_1, \mathscr{L}_N$ models contact with a reservoir at left and right ends of the chain. We call $L_{i,i+1}$ *the bulk single edge generator*. This is actually the crucial object we are after (in some sense the "correct" reservoirs will also be determined once the bulk part of the generator is found). The single site state space K is depending on the models $\{0, 1, \dots, 2j\}$, $\mathbb{N}$ $[0, \infty)$ or $\mathbb{R}$. We will restrict here to systems with a single conserved quantity (particle number or total energy).

The reservoirs are (therefore) characterized by a single parameters (the density, or chemical potential, the temperature) and are such that when the parameters of the two reservoirs are equal the system is in equilibrium, i.e., selects one particular equilibrium state of the one-parameter family of equilibrium measures (stationary measures) of the bulk generator. The models which we want to study are such that

1. The bulk part has a (single) conserved quantity (total number of particles, total energy).

2. The bulk part is self-dual or has a "nice" dual.

3. The full system has a dual where the reservoirs are replaced by absorbing boundaries.

This leads to the following general results

1. The system has a unique stationary state, for equal reservoir parameters this is a stationary measure of the bulk part. For different reservoir parameters this is a so-called non-equilibrium steady state (NESS) which carries a current (transport from one reservoir to the other).

2. The non-equilibrium profile is characterized by a single random walker absorbed at the left or right end. In the symmetric case this implies a linear profile (Fick's law, Fourier law).

3. For the computation of well-chosen n-point correlation functions, we need exactly n dual particles. I.e., quantities of a possible large system can be computed by only a few dual particles ("from many to few"). For $n = 2$ these can be computed (sometimes) analytically. The asymptotic behavior is multi-linear (microscopic multi-linearity is not guaranteed by duality).

Remark 2.1. *For systems with duality or self-duality, we can also easily define* the infinite volume limit*, which would be a problem for processes like the SIP because of unbounded number of particles.*

We start now with an overview of the various *symmetric models*. By symmetric here we mean that the system in equilibrium (for equal reservoir parameters) is *reversible*.

2.1 Models of inclusion type

These models turn up in modelling heat transport (transport of energy), wealth distribution, population genetics, particle transport (particle models where particles attract each other). I will describe the bulk generators (the possible boundary generators are more or less fixed by the requirement that the dual is absorbing).

1. **The symmetric inclusion process. SIP$(2k)$, $k > 0$.**

a) State space $\mathbb{N}^N$ bulk generator

$L_{i,i+1}f(\eta) = \eta_i(2k+\eta_{i+1})(f(\eta^{i,i+1})-f(\eta))+\eta_{i+1}(2k+\eta_i)(f(\eta^{i+1,i})-f(\eta))$

The generator $L_{i,i+1}$ conserves $\eta_i + \eta_{i+1}$, so the bulk process conserves the total number of particles $\sum_{i=1}^N \eta_i$.

b) Stationary (reversible) product measures: products of discrete Gamma distributions with shape parameter $2k$, i.e., with marginals

$$\begin{aligned} \nu_{2k,\lambda}(n) &= (1-\lambda)^{2k}\frac{\lambda^n}{n!}\frac{\Gamma(2k+n)}{\Gamma(2k)} \\ &= \frac{\lambda^n}{Z_\lambda}\binom{2k-1+n}{n} \end{aligned} \tag{2}$$

where $0 < \lambda < 1$, and $Z_\lambda = (1-\lambda)^{-2k}$ is the normalizing constant. Special case $2k = 1$, then these measures are geometric, for $k = m/2$, $m \in \mathbb{N}_0$ they are *negative binomial.* They also satisfy the "addition" property $X \approx \nu_{2k,\lambda}$, $Y \approx \nu_{2k',\lambda}$, $X \perp Y$, then $X+Y \approx \nu_{2k+2k',\lambda}$

c) Natural polynomials (normalized factorial moments) associated to the reversible product measures are

$$d_{2k}(n,m) = \begin{cases} \frac{m!}{(m-n)!}\frac{\Gamma(2k)}{\Gamma(2k+n)} & n \le m \\ 0 \text{ otherwise} \end{cases}$$

The link between these polynomials and the reversible product measures is

$$\sum_{m=0}^{\infty} d_{2k}(n,m)\nu_{2k,\lambda}(m) = \rho(\lambda)^n \tag{3}$$

with $\rho(\lambda) = \frac{\lambda}{1-\lambda}$

d) These polynomials are naturally extended to the multivariate case by

$$\mathcal{D}_{2k}(\xi,\eta) = \prod_{i=1}^N d_{2k}(\xi_i,\eta_i)$$

If we denote by $\sum_{i=1}^n \delta_{x_i}$ a configuration of particles where particles are located at $(x_1,\dots,x_n)$ (with possible repetitions), and $\Lambda : \{1,\dots,N\} \to (0,1)$ denotes a profile for the parameter λ then the relation (3) becomes

$$\int \mathcal{D}_{2k}\left(\sum_{i=1}^n \delta_{x_i},\eta\right) \otimes_{i=1}^N \nu_{2k,\lambda(i)}(d\eta) = \prod_{i=1}^n \rho(\lambda(x_i)) \tag{4}$$

2. **Thermalized SIP**$(2k)$**: discrete redistribution process.**
   Instead of moving one particle at each jump, we "instantaneously" equilibrate edges at the event times of a mean one Poisson process. I.e., every edge has a clock ringing after an exponential (mean one) time after which the masses $(n, m) = (\eta_i, \eta_{i+1})$ are replaced by

$$(n', m + n - n')$$

   where n' has a Beta Binomial distribution with parameters $n+m, 2k, 2k$. This is the distribution of $X|X + Y = n + m$ where X, Y are iid with distribution (2). It is also described by its distribution

$$\mathbb{P}_{n+m,2k,2k}(n' = l) = \binom{n+m}{l} \mathbb{E}p^l(1-p)^{n+m-l}$$

   where $\mathbb{E}$ is expectation over p over the Beta$(2k, 2k)$ distribution. For $2k = 1$ this coincides with the discrete uniform distribution over the set $\{0, 1, \dots, n+m\}$. For this particular case, the redistribution process is also known as *the dual KMP process*. The reversible product measures are of course the same, and so are the natural associated polynomials.

   The single edge generator is given by

$$L_{i,i+1}f(\eta) = \sum_{n'=0}^{\eta_i+\eta_{i+1}} \mathbb{P}_{n+m,2k,2k}(n') \left(f(\eta^{i,i+1;n',\eta_i+\eta_{i+1}-n'}) - f(\eta) \right)$$

   where $\eta^{i,i+1;k,l}$ denotes the configuration obtained from η by replacing η_i with k and η_{i+1} with l.

3. **Diffusion of energy: The Brownian Energy Process (BEP)**$(2k)$

   a) From SIP to BEP:
      Rescaling $\eta_i = \lfloor K z_i \rfloor, i = 1, \dots, N$ and letting evolve $\eta_i, i = 1, \dots, N$ as the SIP$(2k)$ gives that the process $z_i(Nt)$ converges weakly to a diffusion process on $[0, \infty)^N$ with generator $\sum_{i=1}^N L_{i,i+1}$ where

$$L_{i,i+1} = z_i z_{i+1} \partial^2_{i,i+1} - 2k(z_i - z_{i+1})\partial_{i,i+1} \tag{5}$$

      where

$$\partial_{i,i+1} = \partial_{z_i} - \partial_{z_{i+1}}$$

b) Reversible product measures are products of Gamma distributions with shape parameter $2k$, i.e. with marginal probability density

$$\nu_{2k,\lambda}(z) = \frac{z^{2k-1}}{\theta^{2k}\Gamma(2k)} e^{-z/\theta}$$

c) Natural polynomials associated to the product measures are

$$d_{2k}(n,z) = \frac{z^n \Gamma(2k)}{\Gamma(2k+n)} \tag{6}$$

and multivariate

$$\mathcal{D}_{2k}(\xi, z) = \prod_{i=1}^{N} d_{2k}(\xi_i, z_i) \tag{7}$$

d) The relation between the polynomials and the product measures is

$$\int \mathcal{D}_{2k}\left(\sum_{i=1}^{n} \delta_{x_i}, \eta\right) \otimes_{i=1}^{N} \nu_{2k,\theta(i)}(d\eta) = \prod_{i=1}^{n} \theta(x_i) \tag{8}$$

4. **Continuous redistribution model: the thermalized BEP**$(2k)$

   a) Here we consider a Poisson process associated to each edge and on the event times redistribute the mass on that edge as follows: an initial mass (z, z') goes to

   $$((z+z')U, (z+z')(1-U))$$

   with U Beta$(2k, 2k)$ distributed, i.e., with density proportional to $u^{2k-1}(1-u)^{2k-1}$ on $[0,1]$. The single edge generator is

   $$L_{i,i+1}f(z) = \mathbb{E}\left(f\left(z^{i,i+1;(z_i+z_{i+1})U,(z_i+z_{i+1})(1-U)}\right) - f(z)\right)$$

   where $\mathbb{E}$ denotes expectation over U, over the Beta$(2k, 2k)$ distribution.

   b) Because the Beta distribution can be obtained as the distribution of $X/(X+Y)$ with X, Y independent Gamma, the redistribution can also be described as follows: when the clock of the edge $i, i+1$ rings, "thermalize" this edge by running the BEP$(2k)$ for infinite time, this leads to a *product of Gamma distributions, conditioned on the sum.*

c) This model has the same reversible product measures, and the same natural polynomials.

d) For $2k = 1$ the model is the well-known KMP model; in that case U is uniform on $[0, 1]$.

e) The model can also be obtained by rescaling the thermalized SIP via the choice

$$\eta_i = K x_i$$

and letting $K \to \infty$.

Remark 2.2. *A final model which we will not further discuss here, but which appeared as a starting model in [11] is the Brownian Momentum Process (BMP) with single edge generator given by*

$$(x_i \partial_{x_{i+1}} - x_{i+1} \partial_{x_i})^2$$

on the state space $\mathbb{R}^2$*. Now* x_i *is interpreted as momentum at site* i*, and this process conserves the total energy* $x_i^2 + x_{i+1}^2$*, and randomly "rotates" the angle* $\theta_{i,i+1} = \arctan(x_{i+1}/x_i)$*. For the boundary generators one chooses Ornstein Uhlenbeck processes*

$$L_1 = 2T_1 \partial_1^2 - x_1 \partial_1$$

For $z_i = x_i^2$ *we have that* z_i *evolves as the BEP with* $2k = 1/2$*. More generally, we can allow* m *momenta at each vertex and choose the single edge generator*

$$\sum_{\alpha,\beta=1}^{m} (x_{i,\alpha} \partial_{x_{i+1,\beta}} - x_{i+1,\alpha} \partial_{x_{i,\beta}})^2$$

and boundary generator independent Ornstein Uhlenbeck processes

$$L_1 = \sum_{\alpha=1}^{m} 2T_1 \partial_{1,\alpha}^2 - x_{1,\alpha} \partial_{1,\alpha}$$

then $z_i = \sum_{\alpha}^{m} x_{i,\alpha}^2$ *evolves as BEP with* $2k = m/2$*.*

2.2 Models of SEP type: short overview

a) State space $\{0, \mathbb{1}, \ldots, 2j\}^N$, where j is half-integer. Bulk single edge generator:

$$\begin{aligned} & L_{i,i+1} f(\eta) \\ = \; & \eta_i (2j - \eta_{i+1})(f(\eta^{i,i+1}) - f(\eta)) + \eta_{i+1}(2j - \eta_i)(f(\eta^{i+1,i}) - f(\eta)) \end{aligned}$$

$j = 1/2$ corresponds to the standard symmetric exclusion process.

b) Reversible product measures ν_ρ: products of binomials with marginals

$$\nu_\rho(k) = \binom{2j}{k} \rho^k (1-\rho)^{2j-k}$$

c) Special polynomial

$$d_{2j}(k,n) = \frac{\binom{n}{k}}{\binom{2j}{k}}$$

for $0 \leq k \leq n \leq 2j$ (defined to be zero otherwise).

We can then also "thermalize" this model, yielding a model where more particles jump at the same "edge" event (as we did for the SIP).

2.3 Dualities

Let us first give the definition of duality

Definition 2.1. *We say that the Markov processes $\mathbb{X} = X_t, t \geq 0$ on the state space Ω and $\mathbb{Y} = Y_t, t \geq 0$ on the state space Ω' are dual to each other with duality function $D : \Omega' \times \Omega \to \mathbb{R}$ if for all $x \in \Omega$, $y \in \Omega'$ and $t > 0$*

$$\mathbb{E}_x(D(y, X_t))) = \hat{\mathbb{E}}_y(D(Y_t, x)) \tag{9}$$

$\mathbb{E}_x$ (resp. $\hat{\mathbb{E}}_y$) denoting expectation over $\mathbb{X}$ starting at x (resp. $\mathbb{Y}$ starting at y) In case $\mathbb{X} = \mathbb{Y}$ we say that the process is self-dual.

We denote this by $\mathbb{X} \longrightarrow^D \mathbb{Y}$. We then have the following dualities

1. Self-duality of SIP
$$SIP(2k) \longrightarrow^D SIP(2k)$$
with duality functions the "natural polynomials" $\mathcal{D}(\xi, \eta)$.

2. Duality of BEP and SIP
$$BEP(2k) \longrightarrow^D SIP(2k)$$
with duality functions the "natural polynomials" $\mathcal{D}(\xi, z)$.

3. Self-duality of Thermalized SIP with duality functions the "natural polynomials" $\mathcal{D}(\xi, \eta)$.

4. Duality of Thermalized BEP with thermalized SIP with duality functions the "natural polynomials" $\mathcal{D}(\xi, z)$. (for $2k = 1$ this is exactly the duality from the original KMP model).

2.4 Adding boundary reservoirs

The boundary reservoir generators are chosen in such a way that the dual process (which is SIP or thermalized SIP) becomes absorbing (it will have two more lattice sites associated to the absorbing boundaries), and such that in the bulk the duality functions are the same, and at the boundary are "thermalized". We first illustrate this for SIP.

Definition 2.2. *1. The SIP with absorbing boundary conditions is the process $\eta^*(t), t \geq 0$ on $\mathbb{N}^{\{0,\ldots,N+1\}}$ with generator*

$$Lf(\eta) = \sum_{i=1}^{N} L^{SIP}_{i,i+1} + L_{1,0} + L_{N,N+1}$$

with

$$L_{1,0}f(\eta) = a\eta_1(f(\eta^{1,0}) - f(\eta))$$
$$L_{N,N+1}f(\eta) = b\eta_N f(\eta^{N,N+1}) - f(\eta))$$

In this process, every particle (independently) can go from 1 to 0 at rate a, and from N to $N+1$, after which it is absorbed. There is no interaction between absorbed and non-absorbed particles.

2. The SIP with reservoirs has boundary generators

$$\mathscr{L}_1 f(\eta) = \alpha(2k + \eta_1)(f(\eta^{0,1}) - f(\eta)) + \gamma\eta_1(f(\eta^{1,0}) - f(\eta))$$

$$\mathscr{L}_N f(\eta) = \delta(2k + \eta_N)(f(\eta^{N+1,N}) - f(\eta)) + \beta\eta_N(f(\eta^{N,N+1}) - f(\eta))$$

where now $\eta^{0,1}$ means a particle extra at 1. In this process with specific rates, particles are injected or removed at the boundary sites $0, N$.

Remark 2.3. *Notice that the rates at the boundary are "birth" and "death" rates compatible with the stationarity of the product reversible measures.*

We then find that absorbing SIP with absorbing rates $a = \gamma - \alpha$, $b = \beta - \delta$ is dual to the reservoir SIP provided $a = \gamma - \alpha > 0$, $b = \beta - \delta > 0$. As duality function, we obtain

$$A^{\xi_0} B^{\xi_{N+1}} \prod_{i=1}^{N} d_{2k}(\xi_i, \eta_i)$$

with

$$A = \frac{\alpha}{\gamma - \alpha}, B = \frac{\delta}{\beta - \delta}$$

Similarly, one can introduce boundary generators for the BEP, such that the duality functions between BEP + boundaries and absorbing SIP are of the form

$$T_L^{\xi_0} T_R^{\xi_{N+1}} \prod_{i=1}^{N} d_{2k}(\xi_i, x_i)$$

This fixed the boundary generators to be of the form

$$L_1 = T_L(2k\partial_{z_1} + z_1\partial_{z_1}^2) - \frac{1}{2} z_1 \partial_{z_1}$$

2.5 Applications

1. Stationary profile.
2. Uniqueness of the NESS and formula for the stationary expectation of duality function in terms of absorption probabilities.
3. Macroscopic limits.

2.6 Some different redistribution models with duality

1. Wealth distribution model with propensities.
2. Immediate exchange model in econophysics.

3 Lecture 2: The Lie algebraic structure for symmetric models

3.1 Duality relation between operators

The duality between processes reduces often to the duality between their corresponding generators. I.e., two generators L (working on $\mathscr{C}(\Omega)$ and $\hat{L}$ (working on $\mathscr{C}(\Omega')$ are dual with duality function D (notation $\hat{L} \longrightarrow^D L$ if for all $x \in \Omega$, $y \in \Omega'$

$$L^{right} D(y,x) = \hat{L}^{left} D(y,x) \tag{10}$$

where L acts on x, $\hat{L}$ acts on y.

More generally, two operators $A, \hat{A}$ are dual with duality function D if

$$A^{right} D(y,x) = \hat{A}^{left} D(y,x)$$

The following are elementary but useful properties of the relation $\longrightarrow^D$.

Proposition 3.1. *1.* Sums and products. *If $A \longrightarrow^D A'$, $B \longrightarrow^D B'$ then $A+B \longrightarrow^D A'+B'$, $AB \longrightarrow^D B'A'$.*

2. Symmetries. *If $A \longrightarrow^D A'$ and $[A,S]=0$, then $A \longrightarrow^{S^{right}D} A'$. Here $[A,S]=AS-SA$ denotes the commutator. If $A \longrightarrow^D A'$ and $[A',S']=0$ then $A \longrightarrow^{S'^{left}D} A'$.*

3. Cheap duality function. *If Ω is countable, and a reversible measure exists for L, then*

$$L \longrightarrow^D L$$

with

$$D(x,y) = \frac{1}{\mu(x)}\delta_{x,y}$$

Property one shows that the relation $\longrightarrow^D$ turns an algebra of operators into an algebra with opposite (in sign) commutation relations, in other words turns an algebra into its dual (with product ($a*b=b.a$)). Conversely, if one has a set of operators $\{a_i, i\in I\}$ with commutation relations, and a set of operators $\{a_i', i\in I\}$ with opposite commutation relations, then they are candidates for a relation of the type $\longrightarrow^D$. More precisely if we find D such that for all $i\in I$ $a_i \longrightarrow^D a_i'$ then for every elements A in the algebra generated by $\{a_i, i\in I\}$, there exists a corresponding element A' in the algebra generated by $\{a_i', i\in I\}$, such that $A\longrightarrow^D A'$. This means that we have to check only duality for the generators of an algebra in order to conclude duality of the algebras.

Remark 3.1. *If the state space is finite or countable, one can rewrite the (self-)duality relation*

$$L^{left}D(y,x) = \hat{L}^{right}D(y,x)$$

in matrix form

$$LD = D\hat{L}^T$$

where T denotes transposition. This also explains the fact that products appear in reverse order when passing to dual objects: $(AB)^T = B^TA^T$.

3.2 Elementary examples from the Heisenberg algebra

The Heisenberg algebra is generated by two elements $a, a^\dagger$ which satisfy

$$[A, A^\dagger] = I$$

so the dual algebra satisfies $[a, a^\dagger] = -I$. A representation is given by $A = d/dx$, $A^\dagger = x$. A discrete representation of the dual algebra is given by

$$af(n) = nf(n-1), a^\dagger f(n) = f(n+1).$$

A continuous representation of the dual algebra is given by

$$b = x, b^\dagger = \frac{d}{dx}.$$

A continuous representation of the algebra is

$$A^\dagger = x, A = \frac{d}{dx}.$$

The intertwiners between the discrete $a, a^\dagger$ and $A, A^\dagger$ (duality functions) are

$$a \longrightarrow^D A, a^\dagger \longrightarrow^D A^\dagger$$

with $D(n, x) = x^n$, and between $b, b^\dagger$ and $A, A^\dagger$

$$b \longrightarrow^D A, b^\dagger \longrightarrow^D A^\dagger$$

with $D(y, x) = e^{xy}$. Some illustrations:

1. **Wright Fisher and Kingman's coalescent block counting process.**

$$x(1-x)\frac{d^2}{dx^2} \longrightarrow^D a^2 a^\dagger (I - a^\dagger)$$

   The operator

$$a^2 a^\dagger (I - a^\dagger) f(n) = n(n-1)(f(n-1) - f(n))$$

   is the generator of the Kingman coalescent block counting process. So the moment duality between the Wright Fisher diffusion and the Kingman's coalescent follows.

2. **Brownian motion.** Similarly,

$$\frac{1}{2}\frac{d}{dx^2} \longrightarrow^D \frac{1}{2}y^2$$

   and so by exponentiation, one finds the well-known simple formula

$$\mathbb{E}_x e^{W_t y} = e^{\frac{1}{2}y^2 t} e^{xy}$$

   where W_t denotes Brownian motion, and $\mathbb{E}_x$ expectation starting from x.

3. **Independent random walkers.** Working now with a two vertex system, the operator

$$\begin{aligned} L &= -(a_1 - a_2)(a_1^\dagger - a_2^\dagger) f(n_1, n_2) \\ &= n_1(f(n_1 - 1, n_2 + 1) - f(n_1, n_2)) + n_2(f(n_1 + 1, n_2 - 1) - f(n_1, n_2)) \end{aligned}$$

describes independent symmetric continuous-time random walkers. It commutes with $a_1 + a_2, a_1^\dagger + a_2^\dagger$. The reversible measure is product of Poisson, and as a consequence a cheap self-duality function is

$$D_{cheap}(m_1, m_2; n_1, n_2) = n_1! n_2! \delta_{m_1,n_1} \delta_{m_2,n_2}$$

working with $e^{a_1+a_2}$ on the m_1, m_2 variables, using

$$e^{a^\dagger} \delta_{m,n} = \left(e^{a^\dagger} \delta_{m,\cdot} \right)(n) = \frac{(a^\dagger)^{n-m}}{(n-m)!} (\delta_{m,\cdot})(n) = \frac{1}{(n-m)!}$$

gives

$$D(m_1, m_2; n_1, n_2) = \frac{n_1! n_2!}{(n_1 - m_1)!(n_2 - m_2)!}$$

as a self-duality function.

3.3 Casimir element of $\mathscr{U}(SU(2))$ and $\mathscr{U}(SU(1,1))$

If we want to find a bulk single edge generator satisfying self-duality properties, then in view of proposition 3.1 we have to find commuting operators. Conversely, if we find a generator which already by construction has commuting operators, then we have self-duality functions. It is exactly this which we will do, using well-known central elements of a Lie algebra and lifting them to the tensor product (i.e., making an operator working on a configuration space with two vertices).

3.3.1 $SU(2)$

$\mathscr{U}(SU(2))$ is the Lie algebra generated by three elements J^+, J^-, J^0 with commutation relations (Lie brackets)

$$[J^+, J^-] = 2J^0, [J^0, J^\pm] = \pm J^\pm$$

A standard representation is the $2j+1$ dimensional representation given by

$$\begin{aligned} J^+|n> &= (2j - n)|n+1> \\ J^-|n> &= n|n-1> \\ J^0|n> &= (n - j)|n> \end{aligned}$$

on an orthonormal basis $|0>, \ldots, |2j>$

The corresponding representation of the dual algebra is given by the operators working on functions $f : \{0, \ldots, 2j\} \to \mathbb{R}$ by

$$\begin{aligned}
\mathscr{J}^+ f(n) &= (2j-n) f(n+1) \\
\mathscr{J}^- f(n) &= n f(n-1) \\
\mathscr{J}^0 f(n) &= (n-j) f(n)
\end{aligned}$$

An important central element is the Casimir element

$$C = J^- J^+ + J^0 (J^0 + I)$$

where I denotes identity. This element commutes with the generators:

$$[C, J^\pm] = 0, [C, J^0] = 0.$$

E.g.

$$\begin{aligned}
[C, J^-] &= J^-[J^+, J^-] + J^0[J^0, J^-] + [J^0, J^-]J^0 + [J^0, J^-] \\
&= 2J^-J^0 - J^-J^- - J^-J^0 - J^- \\
&= [J^-, J^0] - J^- = J^- - J^- = 0
\end{aligned}$$

and analogously for J^+, J^0.

3.3.2 $SU(1,1)$

$\mathscr{U}(SU(1,1))$ is generated by $K^\pm, K^0$ with commutation relations

$$[K^+, K^-] = -2K^0, [K^0, K^\pm] = \pm K^\pm,$$

so they differ from the previous commutation relation by a sign in the $[K^+, K^-] = -2K^0$ (as opposed to $[J^+, J^-] = 2J^0$ in the $\mathscr{U}(SU(2))$ case). The consequences are however important: this group is not compact and the corresponding algebra has no finite dimensional representations. The models built from these commutation relations turn out to be "bosonic" counterparts of their "fermionic" $\mathscr{U}(SU(2))$ counterparts (inclusion versus exclusion, attractive versus repulsive interaction).

The Casimir element is now given by

$$C = K^0(K^0 - I) - K^+K^-$$

where I denotes identity. Indeed we see once more that C is central, e.g.

$$\begin{aligned}
[C, K^-] &= [K^0, K^-]K^0 + K^0[K^0, K^-] - [K^0, K^-] - [K^+, K^-]K^- \\
&= -K^-K^0 - K^0K^- + K^- + 2K^0K^- \\
&= [K^0, K^-] + K^- = 0
\end{aligned}$$

3.4 Co-product and computation of the co-product of the Casimir in $SU(2)$ and $SU(1,1)$

A co-product on an algebra $\mathscr{A}$ is an algebra homomorphism from $\mathscr{A} \to \mathscr{A} \otimes \mathscr{A}$:

$$\Delta : \mathscr{A} \to \mathscr{A} \otimes \mathscr{A}$$

So it suffices to define it on the generators. Moreover, as a consequence of the homomorphism property, it holds that

$$[A, B] = 0 \text{ implies } [\Delta(A), \Delta(B)] = 0$$

For both $\mathscr{U}(SU(2)), \mathscr{U}(SU(1,1))$ we define the co-products

$$\Delta(J^\alpha) = J_1^\alpha + J_2^\alpha = J^\alpha \otimes I + I \otimes J^\alpha \tag{11}$$

Let us see for $\mathscr{U}(SU(1,1))$ case that this indeed is a "consistent" definition, i.e., we verify e.g.

$$\Delta[K^+, K^-] = [\Delta(K^+), \Delta(K^-)]$$

$$\Delta([K^+, K^-]) = -2\Delta(K^0) = -2(K_1^0 + K_2^0)$$

and

$$\begin{aligned} [\Delta(K^+), \Delta(K^-)] &= [K_1^+ + K_2^+, K_1^- + K_2^-] \\ &= [K_1^+, K_1^-] + [K_2^+, K_2^-] \\ &= -2(K_1^0 + K_2^0) \end{aligned}$$

As a consequence of the relation $[C, A] = 0$ for C the Casimir element, and A every other element of the algebra, we have $[\Delta(C), \Delta(A)] = 0$, and hence in particular, in $\mathscr{U}(SU(2))$ (resp. $\mathscr{U}(SU(1,1))$), $\Delta(C)$ commutes with $J_1^\alpha + J_2^\alpha$ (resp. $K_1^\alpha + K_2^\alpha$).

3.5 From the co-product of the Casimir to the generator

$\mathscr{U}(SU(2))$

Let us now compute, first in $\mathscr{U}(SU(2))$ the co-product of the Casimir.

$$\begin{aligned} & \Delta(J^0(J^0 + I) + J^- J^+) \\ = \ & (J_1^0 + J_2^0)^2 + (J_1^0 + J_2^0) + (J_1^- + J_2^-)(J_1^+ + J_2^+) \\ = \ & (J_1^0)^2 + J_0^1 + (J_2^0)^2 + J_2^0 \\ + \ & 2J_1^0 J_2^0 + J_1^- J_1^+ + J_2^- J_2^+ \\ + \ & J_1^- J_2^+ + J_2^- J_1^+ \\ = \ & C_1 + C_2 + J_1^- J_2^+ + J_2^- J_1^+ + 2J_1^0 J_2^0 \end{aligned}$$

This operator commutes with $J_1^\alpha + J_2^\alpha$, and, therefore, so does

$$H = J_1^- J_2^+ + J_2^- J_1^+ + 2J_1^0 J_2^0$$

because C_1, C_2 also commute with $J_1^\alpha + J_2^\alpha$. Let us now compute this operator H in the concrete representation of the dual algebra. We denote $\eta = \eta_1, \eta_2$ a particle configuration $\eta_i \in \{0, \ldots, 2j\}$, and denote $\eta^{12} = (\eta_1 - 1, \eta_2 + 1)$

$$\begin{aligned} Hf(\eta_1, \eta_2) &= \eta_1(2j - \eta_2)f(\eta - 1, \eta_2 + 1) + (2j - \eta_1)\eta_2 f(\eta_1 + 1, \eta_2 - 1) \\ &+ 2(\eta_1 - j)(\eta_2 - j)f(\eta_1, \eta_2) \\ &+ \eta_1(2j - \eta_2)(f(\eta^{12}) - f(\eta)) + \eta_2(2j - \eta_1)(f(\eta^{21}) - f(\eta)) - 2j^2 f(\eta_1, \eta_2) \end{aligned}$$

denoting the generator

$$Lf(\eta) = \eta_1(2j - \eta_2)(f(\eta^{12}) - f(\eta)) + \eta_2(2j - \eta_1)(f(\eta^{21}) - f(\eta))$$

we see that H equals $L - 2j^2 I$. Hence, the operators that commute with H coincide with those that commute with L.

As a conclusion we followed the following road

1. Start from the central element C.
2. Apply the co-product.
3. Remove Casimirs and constants.
4. Arrive at the generator of a Markov process.

$\mathscr{U}(SU(1,1))$

Let us now show that applying the same procedure in $\mathscr{U}(SU(1,1))$ with a discrete representation of the dual algebra, leads to the SIP.

$$\begin{aligned} & \Delta(K^0(K^0 - I) - K^+K^-) \\ = \; & (K_1^0)^2 - K_1^0 + (K_2^0)^2 - K_2^0 \\ & -K_1^+K_1^- - K_2^+K_2^- + 2K_1^0K_2^0 - K_2^+K_1^- - K_1^-K_2^+ \\ = \; & C_1 + C_2 - (-2K_1^0K_2^0 + K_2^+K_1^- + K_1^-K_2^+) \end{aligned}$$

From this computation, we conclude that

$$H = -2K_1^0K_2^0 + K_2^+K_1^- + K_1^-K_2^+$$

commutes with $K_1^\alpha + K_2^\alpha$. Now consider the representation

$$K^+ f(n) = (2k + n)f(n + 1) \tag{12}$$
$$K^- f(n) = nf(n - 1) \tag{13}$$
$$K^0 f(n) = (k + n)f(n) \tag{14}$$

Computing H then gives $H = L - 2k^2 I$ with

$$L = \eta_1(2k + \eta_2)(f(\eta^{12}) - f(\eta)) + \eta_2(2k + \eta_1)(f(\eta^{21}) - f(\eta))$$

which is the generator of the $SIP(2k)$.

3.5.1 Co-associativity of the co-product

The co-product can be defined on multiple tensor products by the co-associativity property which is

$$(I \otimes \Delta)(\Delta(A)) = (\Delta \otimes I)(\Delta(A))$$

Indeed, e.g. in the $SU(2)$ case with $A = J^+$ say, we compute, using $\Delta(I) = I \otimes I$

$$\begin{aligned}(I \otimes \Delta)(\Delta(J^+)) &= (I \otimes \Delta)(J^+ \otimes I + I \otimes J^+) \\ &= J^+ \otimes I \otimes I + I \otimes J^+ \otimes I + I \otimes I \otimes J^+\end{aligned}$$

and $(\Delta \otimes I)(\Delta(J^+))$ yields the same. Therefore we can define the n-fold co-product $\Delta^{(n)} : \mathscr{A} \to \mathscr{A} \otimes \ldots \otimes \mathscr{A}$ inductively via

$$\Delta^{(n)} = \Delta^{(n-1)} \otimes I$$

which e.g. in the $SU(2)$ case gives

$$\Delta^{(n)}(J^\alpha) = \sum_{i=1}^{n} J_i^\alpha$$

3.6 Consequences of the construction: self-duality of SEP and SIP

Let us start with the $\mathscr{U}(SU(2))$ case. We have arrived now at a generator which by construction commutes with $J_1^\alpha + J_2^\alpha$.

We will now act with one of the symmetries on the η-variable of the cheap duality function

$$D(\xi, \eta) = \frac{1}{\binom{2j}{\xi_i}} \delta_{\eta_i, \xi_i}$$

Notice that for $l \geq m$ we have

$$(J^-)^k \delta_{l,m} = \begin{cases} l(l-1)\ldots(m+1) = (l-m)!\binom{l}{m} \text{ if } k = (l-m) \\ 0 \text{ otherwise} \end{cases}$$

where $(J^-)^k$ works on the l-variable. As a consequence

$$e^{J_1^-}\delta_{l,m} = \frac{(J^-)^{l-m}}{(l-m)!}\delta_{l,m} = \binom{l}{m}$$

As a further consequence

$$e^{J_1^- + J_2^-} D(\xi,\eta)(\text{ working on } \eta)$$

equals

$$\mathcal{D}(\xi,\eta) = \prod_i \frac{\binom{\eta_i}{\xi_i}}{\binom{2j}{\xi_i}}$$

and thus we find that this is a self-duality function for the SEP$(2j)$.

Similarly, we find the self-duality function of the SIP$(2k)$ by applying the symmetry $e^{K_1^- + K_2^-}$ (working on the η variable) to the cheap duality function.

Remark 3.2. *We gave now the procedure to produce a generator for a two site system. We then just copy this on the nearest neighbor edges to find (in $\mathscr{U}(SU(2))$ case and analogously in the $\mathscr{U}(SU(1,1))$ case, as a starting operator*

$$\sum_{i=1}^{N} [\Delta(C)]_{i,i+1}$$

which then commutes with $\sum J_i^\alpha$. Indeed

$$[\Delta(C)]_{i,i+1} = I \otimes I \otimes \cdots (C_i + C_{i+1}) \otimes I \otimes \ldots \otimes I$$

commutes automatically with J_k^α, $k \notin \{i, i+1\}$. Therefore

$$\left[\sum_{i=1}^{N} [\Delta(C)]_{i,i+1}, \sum_{i=1}^{N} J_i^\alpha\right] = \sum_i \left[[\Delta(C)]_{i,i+1}, J_i^\alpha + J_{i+1}^\alpha\right] = 0$$

4 Lecture 3: The ASIP(q,k) and its limits

4.1 From a Hamiltonian to a generator: ground state transformation

The procedure to start from a central element and lift it to the tensor product by a co-product is very general, but does not necessarily give a Markov generator. In the symmetric case, we have seen that we had a Markov generator

+ central elements. In the asymmetric case (where co-products are more involved) the procedure usually yields a Schrödinger operator, i.e., a generator plus a multiplication operator. Here we first review how from such a operator a Markov process can be obtained, provided a non-negative ground state exists. This is a variation on the theme how from the harmonic oscillator Hamiltonian one arrives at the generator of the Ornstein Uhlenbeck process.

In general, if L is a Markov generator and e^{ψ} is in the domain, and such that $e^{\psi}g$ is in the domain for sufficiently many g in the domain, then

$$L_{\psi}(g) := e^{-\psi}L(e^{\psi}g) - (e^{-\psi}L(e^{\psi}))g$$

is a generator. Examples

1. If L is of the form of a discrete jump process (with rates c) generator on a finite state space,

$$Lg(x) = \sum_{y} c(x,y)(f(y) - f(x))$$

   then

$$L_{\psi} = \sum_{y} c_{\psi}(x,y)(f(y) - f(x))$$

   with modified rates $c_{\psi}(x,y) = c(x,y)e^{\psi(y)-\psi(x)}$.

2. If $L = \frac{1}{2}\frac{d^2}{dx^2}$ then with $e^{\psi} = e^{-x^2/2}$ we find

$$L_{\psi} = -x\frac{d}{dx} + \frac{1}{2}\frac{d^2}{dx^2}$$

   the Ornstein Uhlenbeck process.

As a consequence we have the following useful (somewhat informally stated) proposition

Proposition 4.1. *Let H be an operator of the form*

$$H(g) = Lg + fg \tag{15}$$

with f a function and L a Markov generator. Assume there exists a function of the form e^{ψ} (i.e., a positive function) such that

$$H(e^{\psi}) = 0$$

(so-called positive ground state). Then

$$L_{\psi}(g) = e^{-\psi}H(e^{\psi}g)$$

is a Markov generator. Moreover, there is a one-to-one correspondence between operators S commuting with H and corresponding operators commuting with L_ψ via

$$S \to S_\psi = e^{-\psi} S(e^{\psi} \cdot)$$

Proof.

$$[L_\psi, S_\psi] = e^{-\psi}[H, S]e^{\psi}$$

□

The condition that H is of the form (15) means in the discrete finite setting that the matrix H has non-positive off-diagonal elements.

4.2 The q-deformed Lie algebra $SU_q(1,1)$, co-product

We now follow the scheme which we followed in the symmetric case but for the algebra $\mathscr{U}_q(SU(1,1))$ with commutation relations

$$[K^0, K^{\pm}] = \pm K^{\pm}, [K^+, K^-] = -[2K^0]_q,$$

where

$$[2K^0]_q = \frac{q^{2K^0} - q^{-2K^0}}{q - q^{-1}}$$

$0 < q < 1$. Notice that for $q = e^{-\epsilon} \approx 1 - \epsilon$

$$[2K^0]_q \approx \frac{e^{-\epsilon 2K_0} - e^{\epsilon 2K_0}}{e^{-\epsilon} - e^{\epsilon}} \approx 2K_0$$

so that in the limit $q \to 1$ we find the classical $\mathscr{U}(SU(1,1))$ commutation relations. Notice that $\mathscr{K} := q^{K_0}$ is formal notation for an element of the algebra with inverse $\mathscr{K}^{-1} = q^{-K_0}$, and one can rewrite the commutation relations in the equivalent way

$$\begin{aligned}
\mathscr{K}\mathscr{K}^{-1} &= \mathscr{K}^{-1}\mathscr{K} = I \\
\mathscr{K}K^+ &= qK^+\mathscr{K} \\
\mathscr{K}K^- &= q^{-1}K^-\mathscr{K} \\
[K^+, K^-] &= -\frac{\mathscr{K}^2 - \mathscr{K}^{-2}}{q - q^{-1}}
\end{aligned}$$

We prefer to go on with the (more suggestive) notation q^{K_0} The Casimir element is now

$$C = [K^0]_q[K^0 - 1]_q - K^+K^- \tag{16}$$

and the co-product is given by

$$\Delta(K^{\pm}) = K^{\pm} \otimes q^{-K_0} + q^{K_0} \otimes K^{\pm} \tag{17}$$
$$\Delta(K^0) = K^0 \otimes I + I \otimes K^0 \tag{18}$$

4.3 The ASIP(q,k)

Applying the co-product to the Casimir leads to a Hamiltonian of the form discussed in section 4.1. A trivial ground state is the vector $\otimes_{i=1}^N 0 >$. This groundstate is however not positive and will be turned into a positive ground state by an "exponential" symmetry (see Gioia's talk for the complete explanation).

The process that comes out has the following single edge bulk generator

$$\begin{aligned} Lf(\eta) &= q^{\eta_i-\eta_{i+1}+(2k-1)}[\eta_i]_q[2k+\eta_{i+1}]_q(f(\eta^{i,i+1})-f(\eta)) \\ &+ q^{\eta_i-\eta_{i+1}-(2k-1)}[\eta_{i+1}]_q[2k+\eta_i]_q(f(\eta^{i+1,i})-f(\eta) \end{aligned} \tag{19}$$

4.3.1 Basic properties

1. **Reversible profile product measures.**

$$\mathbb{P}^\alpha(\eta_i = n) = \alpha^n \binom{n+2k-1}{n}_q q^{4kin}$$

$1 \le i \le N$, $\alpha \in [0, q^{-(2k+1)})$ where the q-numbers are defined via

$$[n]_q = \frac{q^n - q^{-n}}{q-q^{-1}}$$

e.g.

$$[0]_q = 0, [1]_q = 2, [2]_q = q+q^{-1}, [3]_q = q^2+1+q^{-2}, \dots$$
$$[n]_q! = [n]_q[n-1]_q \dots [1]_q$$

(there is a suitable generalization for non-integer n given via the q-Gamma function) and the q-deformed binomial coefficient (cf. Koekoek and Swartouw)

$$\binom{n}{m}_q = \frac{[n]_q!}{[m]_q![n-m]_q!}$$

These measures have density profile

$$\mathbb{E}^\alpha(\eta_i) = \sum_{l=0}^{2k-1} \frac{1}{q^{-2l}(\alpha q^{4ki-2k+1})^{-1}-1}$$

which is a decreasing function of i, with $\lim_{i\to\infty} \mathbb{E}^\alpha(\eta_i) = 0$.

2. **Self-duality** The ASIP(q,k) is self-dual with self-duality functions defined as follows.

$$N_i(\eta) = \sum_{k=i}^N \eta_k$$

Notice that

$$N_i(\eta_t) - N_i(\eta_0) = J_i(t)$$

the total integrated current over the edge $(i, i+1)$.

The duality function is best explained first for a single particle dual configuration δ_l (single particle at site l).

$$\mathcal{D}(\delta_l, \eta) = \frac{q^{-4kl+1}}{q^{2k} - q^{-2k}} \left(q^{2N_l}(\eta) - q^{2N_{l+1}}(\eta) \right)$$

So we have

$$\mathbb{E}_l \mathcal{D}(\delta_{l(t)}, \eta) = \mathbb{E}_\eta \frac{q^{-4kl+1}}{q^{2k} - q^{-2k}} \left(q^{2N_l(\eta_t)} - q^{2N_{l+1}(\eta_t)} \right) \tag{20}$$

The l.h.s. of (20) is an expectation with respect to *a single ASIP*(q,k) *particle, which is a simple asymmetric random walk jumping to the left at rate* $q^{-2k}[2k]_q$ *and to the right at rate* $q^{2k}[2k]_q$ *(drift to the left)*. For a more general dual configuration, with n dual particles *at different locations* $l_1, \ldots, l_n$ we get

$$\mathcal{D}\left(\sum_{i=1}^n \delta_{l_i}, \eta\right) = \frac{q^{-4k\sum_{m=1}^n l_m} q^{-n^2}}{(q^{2k} - q^{-2k})^n} \prod_{m=1}^n \left(q^{2N_{l_m}}(\eta) - q^{2N_{l_m+1}}(\eta) \right)$$

4.4 Application: computation of q-moments of current

Let us consider one dual particle and start again from (20), and multiply both sides with $q^{-2N_l(\eta)}$. We get

$$\begin{aligned} \mathbb{E}_\eta q^{2J_l(t)} &= q^{-2\eta_l} \mathbb{E}_\eta q^{2J_{l+1}(t)} \\ &+ q^{4kl} \mathbb{E}_l \left(q^{-4kl(t)} q^{2(N_{l(t)}(\eta) - N_l(\eta))} - q^{2(N_{l(t)+1}(\eta) - N_l(\eta))} \right) \end{aligned}$$

Iterating this we get

$$\mathbb{E}_\eta q^{2J_l(t)} = q^{2(N(\eta) - N_i(\eta))} - \sum_{n=-\infty}^{l-1} q^{4kn} \mathbb{E}_n q^{-4kn(t)} (1 - q^{-2\eta_{n(t)}}) q^{2N_{n(t)}(\eta) - N_l(\eta)} \tag{21}$$

where by definition $2(N(\eta) - N_i(\eta)) = \sum_{n<i} \eta_n$ (which can be infinite but $q^\infty = 0$). This relation has to be thought of as the analogue of the simple expected number of particles formula in the symmetric case, where expected number of particles can be obtained from the starting configuration and a single symmetric random walker. Here it is the q-moment of the current which can be obtained using a single asymmetric random walker.

4.5 The diffusion limit $q = 1 - \frac{\sigma}{N}$, $\eta_i = z_i N$: ABEP(σ, k)

We will now take the limit where $q = 1 - \frac{\sigma}{N} \to 1$ and $\eta_i = \lfloor x_i N \rfloor$ and scale up time by a factor N, i.e. consider the process $\{x_i(Nt), t \geq 0\}$. Then Taylor expansion shows that

$$\lim_{N\to\infty} L_{i,i+1}^{ASIP\left(1-\frac{\sigma}{N},k\right)} F(x_i, x_{i+1}) = L_{i,i+1}^{ABEP(\sigma,k)} F(x_i, x_{i+1})$$

where

$$\begin{aligned} L_{ABEP(\sigma,k)} &= \frac{1}{4\sigma^2}(1 - e^{-2\sigma x_i})(e^{2\sigma x_{i+1}} - 1)(\partial_{x_i} - \partial_{x_{i+1}})^2 \\ - &\frac{1}{2\sigma}\left((1 - e^{-2\sigma x_i})(e^{2\sigma x_{i+1}} - 1) + 2k(2 - e^{-2\sigma x_i} - e^{2\sigma x_{i+1}})\right)(\partial_{x_i} - \partial_{x_{i+1}}) \end{aligned}$$

This process can be compared to the Wright Fisher diffusion with mutation (rate k) and selection (rate σ). Indeed, if σ is very small, this generator becomes to first order in σ

$$x_i x_{i+1} \partial_{i,i+1}^2 - (2\sigma x_i x_{i+1} + 2k(x_i - x_{i+1}))\partial_{i,i+1}$$

As a consequence of self-duality we "loose" the asymmetry for a finite number of dual particles, and therefore the ABEP(σ, k) is dual to the SIP(k), i.e., a truly "asymmetric" process has a symmetric dual. The asymmetry σ is hidden in the duality functions

$$D^\sigma(\xi, x) = \prod_i \frac{\Gamma(2k)}{\Gamma(2k + \xi_i)} \frac{e^{-2\sigma E_{i+1}(x)} - e^{-2\sigma E_i(x)}}{2\sigma} \tag{22}$$

where

$$E_i(x) = \sum_{k=i}^{N} x_k$$

In particular $E_i(x(t)) - E_i(x(0)$ is the integrated energy current over the edge $(i-1, i)$. A computation such as we did for the ASIP can now be done, using that the dual walker is now a symmetric continuous-time symmetric nearest neighbor random walk jumping at rate $2k$, for which we have the explicit formula

$$\mathbb{P}_i(l(t) = n) = e^{-4kt} I_{|n-i|}(4kt)$$

We get

$$\mathbb{E}_x^{ABEP(\sigma,k)} e^{-2\sigma J_i(x(t))} = e^{-4kt} \sum_{n\in\mathbb{Z}} e^{-2\sigma(E_n(x)-E_i(x))} I_{|n-i|}(4kt)$$

Notice in the limit $\sigma \to 0$ this becomes the Beta$(2k-1, 2k-1)$ distribution.

4.6 The Asymmetric KMP process

Just as we thermalized the BEP process, obtaining thereby an energy redistribution process which is a generalization of the KMP process, we can thermalize the ABEP, obtaining an asymmetric energy redistribution model, with the same discrete dual, i.e., the thermalized SIP$(2k)$. In this process we get the following stochastic redistribution rule:

$$(x,y) \to (B_\sigma^{(x+y)}(x+y), (1-B_\sigma^{(x+y)})(x+y))$$

where $B_\sigma^{(E)}$ is a $[0,1]$ random variable with density E dependent density given by

$$\nu_{\sigma,k}(w|E) = Ce^{2\sigma Ew}\left((e^{2\sigma Ew}-1)(1-e^{-2\sigma E(1-w)})\right)^{2k-1}$$

For the choice $2k=1$ this yields the "correct" asymmetric analogue of the KMP process.

4.7 The ABEP(q,k) as a non-local transformation of the BEP(k)

Given that the ABEP is dual to a *symmetric process* (the SIP), one can ask for its corresponding algebraic structure. The answer is that the symmetry of the BEP is again (undeformed) $SU(1,1)$ but with a "conjugate" representation. This is the content of the following proposition.

Proposition 4.2. *Define*

$$E_i(x) = \sum_{l=i}^{N} x_i$$

with $E_{N+1}(x) = 0$ *by definition. Next define*

$$g_i(x) = \frac{e^{-2\sigma E_{i+1}(x)} - e^{-2\sigma E_i(x)}}{2\sigma}$$

Then, if $X(t) = x_i(t), i = 1, \ldots, N$ *evolves according to the ABEP*$(si, 2k)$, *then*

$$Z(t) = g(X(t))$$

evolves according to BEP$(2k)$.

4.8 Open issues

1. Multitype particle systems and Onsager symmetries.

2. Link with Matrix ansatz, multilinearity, Bethe ansatz.

3. Hydrodynamic limits for weakly asymmetric models of SIP type.

Acknowledgment

This work has been done in collaboration with many people G. Carinci (Delft), C. Giardin'a (Modena), C. Giberti (Modena), J. Kurchan (Paris), T. Sasamoto (Tokyo), K. Vafayi (Eindhoven). On the Lie algebraic subjects we received important input from E. Koelink (Nijmegen), W. Groenevelt (Delft).

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