# Disorder-induced traveling waves in the quenched Kuramoto model

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Joint work with Christophe Poquet (Roma $\to$ Lyon)

[Giacomin, L., Poquet, 2014]

[L., Poquet, 2015, arXiv :1505.00497]

# The stochastic Kuramoto model

We consider the system of $N$ stochastic differential equations

$$\mathrm{d}\theta_{i,t} = \omega_i \, \mathrm{d}t + \frac{K}{N} \sum_{j=1}^{N} \sin(\theta_{j,t} - \theta_{i,t}) \, \mathrm{d}t + \mathrm{d}B_{i,t}, \quad i = 1, \cdots, N,$$

- $\theta_i \in \mathbb{S} := \mathbb{R}/2\pi$ (phase oscillators),
- $K > 0$ : interaction intensity
- $\{B_i\}_i$ : i.i.d. standard Brownian motions (thermal noise).
- $\{\omega_i\}_i$ : i.i.d. $\sim \lambda$ (local frequency of the particles, random environment).

## Remarks

- Invariance by rotation : if $\{\theta_j(t)\}_{j=1\dots N}$ is solution, so is $\{\theta_j(t) + \psi\}_{j=1\dots N}$, for all $\psi \in \mathbb{S}$.
- When $\omega_i \equiv 0$, the process is reversible under the invariant measure $\pi_{N,K}$ (Hamiltonian Mean-Field model, HMF or XY model)

$$\pi_{N,K}(\,\mathrm{d}\theta) \propto \exp\left(\frac{K}{N} \sum_{i,j=1}^{N} \cos(\theta_i - \theta_j)\right) \mathrm{d}\theta$$

# The (reversible) case without disorder : $\omega_i \equiv 0$.

$$d\theta_{i,t} = \frac{K}{N} \sum_{j=1}^{N} \sin(\theta_{j,t} - \theta_{i,t})\, dt + dB_{i,t}, \quad i = 1, \cdots, N,$$

If $K$ is sufficiently large, the dynamics leads to the synchronization of the particles along a fixed nontrivial stationary density, at least on bounded time intervals $[0, T]$.

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## Remark

L. Bertini, G. Giacomin, C. Poquet [✎ PTRF 2014] have shown that, when one looks at time scale of order $N$, the center of synchronization performs a Brownian motion on the circle $\mathbb{S}$.

# Adding disorder : traveling waves

$$\mathrm{d}\theta_{i,t} = \omega_i \,\mathrm{d}t + \frac{K}{N}\sum_{j=1}^{N} \sin(\theta_{j,t} - \theta_{i,t})\,\mathrm{d}t + \mathrm{d}B_{i,t}, \quad i = 1, \cdots, N,$$

Competition between the mean-field term ($\rightarrow$ synchronization) and the disorder will induce traveling waves :
Set $\omega_i = \bar{\omega}$ for all $i = 1, \ldots, N$. Then, the synchronized state will rotate with speed $\bar{\omega}$.
More generally, by the change of variables

$$\theta_{i,t} \to \theta_{i,t} - \mathbb{E}_\lambda(\omega)t,$$

one can always assume that the law of the disorder is centered

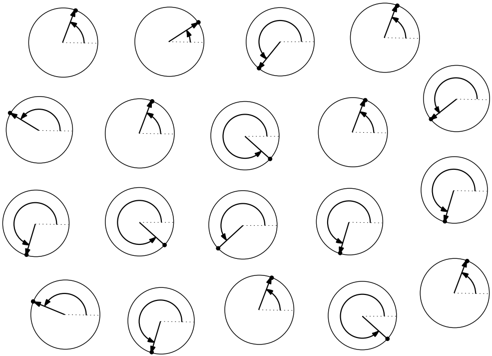
$$\mathbb{E}_\lambda(\omega) = 0.$$

## Questions

- What is the influence of the disorder on the existence of traveling waves ?
- Does it depend only on the law $\lambda$ or on a typical realisation of $(\omega_i)$ ?
- On which time scale do the traveling waves appear ?

# With disorder : the asymmetric case

Suppose that we choose the frequencies $\omega_i$ according to $\lambda = p\delta_{-(1-p)} + (1-p)\delta_p$ with $p < 1/2$.



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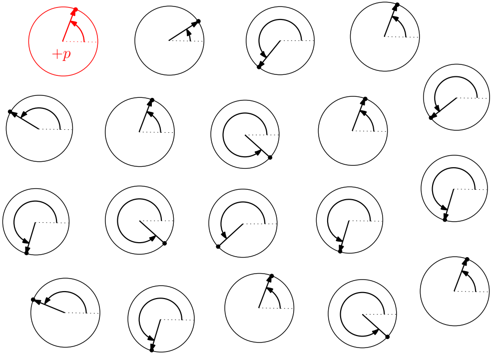

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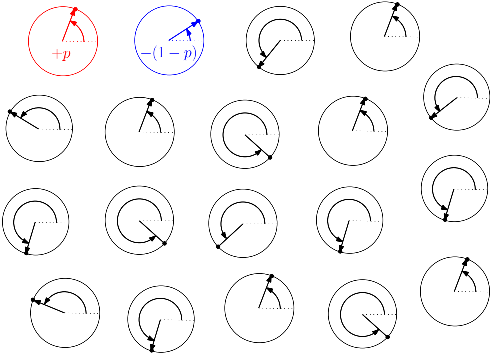

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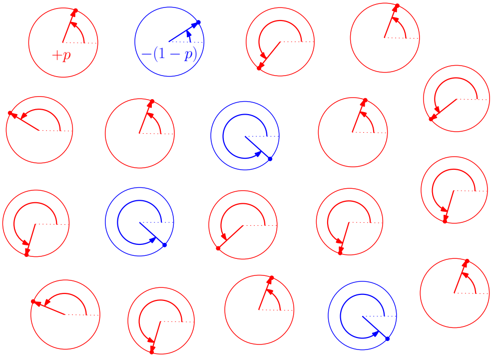

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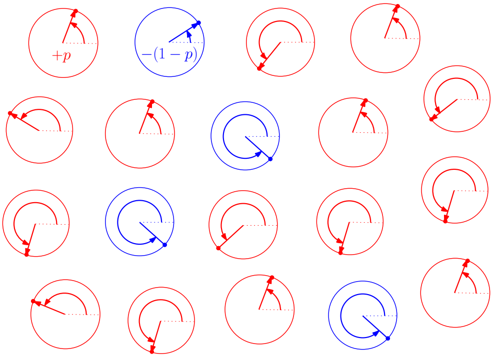
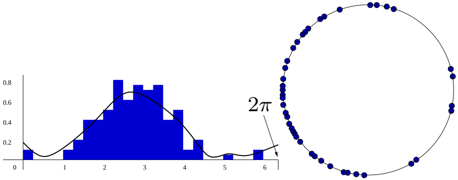



- Who wins ? The majority of red rotators with low frequency or the minority of blue rotators with large frequency ?
- This macroscopic asymmetry is essentially due to a law of large numbers for the disorder $(\omega_i)_i$ : one should see deterministic traveling waves on bounded time intervals $[0, T]$.

The (disordered) empirical measure of the system is given by

$$\mu_{N,t} := \frac{1}{N}\sum_{j=1}^{N}\delta_{(\theta_{j,t},\omega_j)},\ t \geqslant 0$$




## Large population limit on bounded time scale $[0, T]$

When $\mathbb{E}_\lambda(|\omega|) < +\infty$, a.s. w.r.t $(\omega_i)_{i\geqslant 1}$, $\mu_N$ converges as $N \to \infty$ in $\mathcal{C}([0,T], \mathcal{M}_1(\mathbb{S}\times\mathbb{R}))$ to $\mu_t(\,\mathrm{d}\theta,\,\mathrm{d}\omega) = p_t(\theta,\omega)\,\mathrm{d}\theta\lambda(\,\mathrm{d}\omega)$, where $p_t$ solves the mean-field equation

$$\partial_t p_t(\theta,\omega) = \frac{1}{2}\partial_\theta^2 p_t(\theta,\omega) - \partial_\theta\left(p_t(\theta,\omega)\left(\omega - K\int \sin(\cdot) * p_t(\cdot,\tilde\omega)\lambda(\,\mathrm{d}\tilde\omega)\right)\right).$$

$p_t(\theta,\omega)$ : density of rotators with phase $\theta$ and frequency $\omega$, in the limit of an infinite population.

# The asymmetric case

Suppose that the support of $\lambda$ is included in $[-\delta, \delta]$, for some $\delta > 0$.

## Theorem (Giacomin, L., Poquet, 2014)

*For every $K > 1$, there exist $\delta_0 = \delta_0(K) > 0$, such that for all $0 < \delta < \delta_0$, there exist $q_\delta \in L^2(\ell \otimes \lambda)$ and $c_\lambda(\delta) \in \mathbb{R}$ such that*

$$(t, \theta, \omega) \mapsto q_\delta(\theta - c_\lambda(\delta)t, \omega)$$

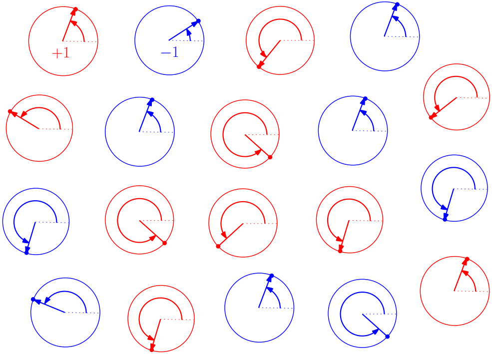
*is a solution of the mean-field equation. Moreover, this family of solutions is stable by perturbation.*

- The speed of rotation $c_\lambda(\delta)$ depends only on the law of the disorder $\lambda$, not on a realization of the disorder.
- If $\lambda$ is symmetric, $c_\lambda(\delta) = 0$ : the mean-field limit on $[0, T]$ does not explain anything on the existence of traveling waves in the symmetric case.
- The proof of the theorem relies on PDE techniques and perturbation of dynamical systems in infinite dimension.

In the case $\lambda = p\delta_{-(1-p)} + (1-p)\delta_p$ with $p < 1/2$, $c_\lambda(\delta) > 0$ : Red wins !

# With disorder : the symmetric case

Now, the disorder $(\omega_i)_{i \geqslant 1}$ is chosen according to $\lambda = \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1$.



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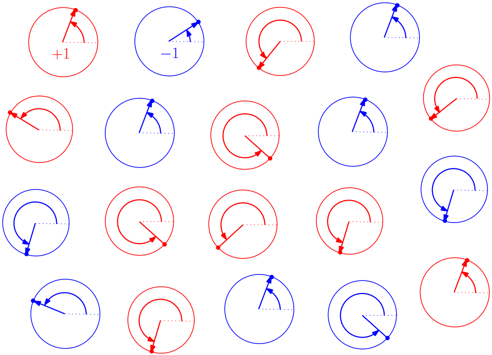
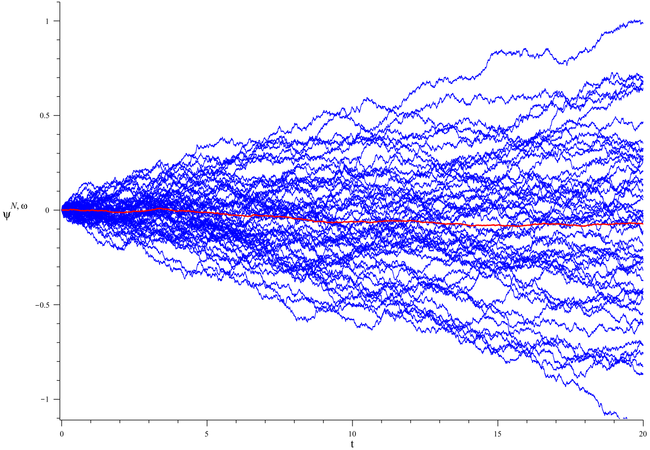



- Here, the law $\lambda$ is symmetric : in the limit as $N \to \infty$, there is no asymmetry in the size of the two populations.
- But for finite (but large) population, finite-size fluctuations of the sample $(\omega_1, \ldots, \omega_N)$ leads to a microscopic asymmetry in the size of the two populations, of order $\sqrt{N}$.
- One expects (random) quenched traveling waves on a time scale of order $\sqrt{N}$.

# Simulation in the symmetric case

# Traveling waves in the symmetric case : center of synchronization




Each trajectory corresponds to a typical sample of the disorder $(\omega_1, \ldots, \omega_N)$.

# The symmetric case : first attempt

Since the traveling waves are due to the fluctuations of the disorder, one should see something at the scale of the CLT of $\mu_N$ around its limit $q$.

## Theorem (L., 2014)

*There exists $H$ Hilbert space, such that on each bounded time interval $[0, T]$, the fluctuation process*

$$\eta_N : t \mapsto \sqrt{N}(\mu_{N,t} - q_t),$$

*converges as $N \to \infty$ (in a weak sense) in $\mathcal{C}([0, T], H)$ to the solution $t \mapsto \eta_t$ of a linear SPDE*

$$\partial_t \eta_t = L_{q_t} \eta_t + \dot{W}_t.$$

*Moreover, as $t \to \infty$, there exists a nontrivial $\ell \in H$, such that*

$$\frac{\eta_t}{t} \to \ell, \quad \text{as } t \to \infty.$$

An informal consequence is that the center of synchronization can be written as

$$\Psi_{N,t} \approx \Psi_t + O\left(\frac{t}{\sqrt{N}}\right).$$

But this is only formal since the limit as $N \to \infty$ is not uniform in $T$ !

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# The symmetric case : second attempt

Suppose for simplicity that $\lambda = \frac{1}{2}(\delta_{-\omega_0} + \delta_{\omega_0})$ for some $\omega_0 > 0$.
Fix $N \geqslant 1$ and denote $N^+$ and $N^-$ the (random) number of rotators possessing the disorder $+\omega_0$ and $-\omega_0$, respectively, and denote by $(\theta^{\pm}_{j,t})_{j=1,\dots,N^{\pm}}$ these rotators, solutions to

$$d\theta^+_{j,t} = +\omega_0 + \frac{K}{N}\left(\sum_{l=1}^{N^+}\sin(\theta^+_{l,t} - \theta^+_{j,t}) + \sum_{l=1}^{N^-}\sin(\theta^-_{l,t} - \theta^+_{j,t})\right)dt + dB_{j,t},$$

$$d\theta^-_{j,t} = -\omega_0 + \frac{K}{N}\left(\sum_{l=1}^{N^+}\sin(\theta^+_{l,t} - \theta^-_{j,t}) + \sum_{l=1}^{N^-}\sin(\theta^-_{l,t} - \theta^-_{j,t})\right)dt + dB_{j,t}$$

The empirical measure $\mu_N$ is here identified with $(\mu^+_N, \mu^-_N)$ defined by

$$\mu^{\pm}_{N,t} = \frac{1}{N^{\pm}}\sum_{j=1}^{N^{\pm}}\delta_{\theta^{\pm}_{j,t}}.$$

With this reformulation, the randomness of the disorder lies in the random sizes $(N^+, N^-)$ of the subpopulations.

The mean-field limit is identified with $(p^+, p^-)$ such that

$$\partial_t p_t^{\pm}(\theta) = \frac{1}{2}\partial_\theta^2 p_t^{\pm}(\theta) - \partial_\theta \left( p_t^{\pm}(\theta) \left( \pm\omega_0 - K \sin * \left( \frac{p_t^+ + p_t^-}{2} \right)(\theta) \right) \right).$$

- $q(\theta) = (\frac{1}{2\pi}, \frac{1}{2\pi})$ is always a stationary solution.
- if $K > 1$, there exists $\omega_0(K)$ such that for all $\omega_0 \leqslant \omega_0(K)$, the mean-field equation admits a unique manifold of synchronized solutions

$$M = \{q_\psi(\cdot), \psi \in \mathbb{S}\},$$

where $q_\psi(\cdot) = q_0(\cdot - \psi)$.

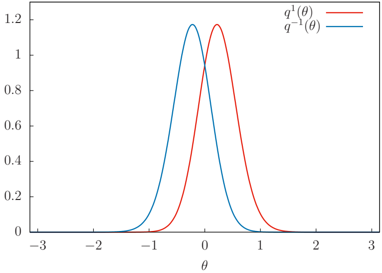


FIGURE – $q_0 = (q^-, q^+)$

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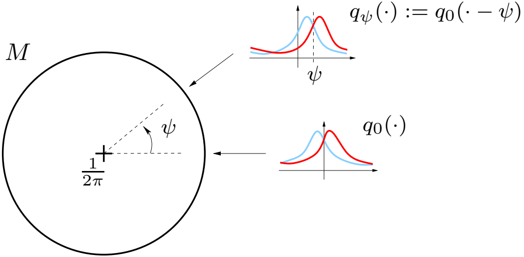


# Local stability of the stationary manifold $M$

Let $q = (q^+, q^-)$ be an element of the stationary manifold $M$ and add a small perturbation $u_t = (u_t^+, u_t^-)$, with $\int_{\mathbb{S}} u_t^\pm(\theta)\,\mathrm{d}\theta = 0$. Then one can easily show that the evolution of $u_t$ is governed by

$$\partial_t u_t = L_q u_t + R_q(u_t),$$

where

$$L_q u_t^\pm = \frac{1}{2}\partial_\theta^2 u_t^\pm - \partial_\theta \left( \pm\omega_0 u_t^\pm - u_t^\pm K \sin * \left( \frac{q^+ + q^-}{2} \right) - q^\pm K \sin * \left( \frac{u_t^+ + u_t^-}{2} \right) \right),$$

is the linearized operator around $q$ and $R_q$ is quadratic.
The evolution in the neighborhood of $q \in M$ is determined by the spectrum of $L_q$.

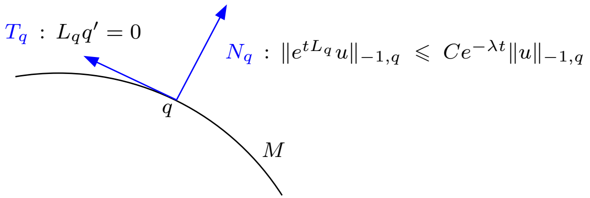
## Definition

For $q \in M$, let $H_q^{-1}$ be the dual of the space $H_q^1$, closure of regular functions $(u^+, u^-)$ with zero mean value on $\mathbb{S}$ under the norm

$$\|u\|_{1,q} := \left( \frac{1}{2} \sum_{\sigma=\pm} \int_{\mathbb{S}} (\partial_\theta u^\sigma(\theta))^2 q^\sigma(\theta)\, \mathrm{d}\theta \right)^{\frac{1}{2}}.$$

## Theorem [Bertini, Giacomin, Pakdaman, 2010], [Giacomin, L., Poquet, 2014] :

For $q \in M$ and $\omega_0$ small enough, $L_q$ can be decomposed in $H_q^{-1}$ as follows :




Moreover, there is a well-defined projection on $M$ adapted to $T_q \oplus M_q$ for $u$ in a suitable neighborhood of $M$.

The neutral direction along $T_q$ reflects the rotational invariance of the system.

# Long-time quenched traveling waves

## Definition :

A sequence of disorder $(\omega_i)_{i \geqslant 1}$ is said to be *admissible* if for all $\zeta > 0$, there exists $N_0$ such that for all $N \geqslant N_0$ such that $\max(\left|\xi_N^+\right|, \left|\xi_N^-\right|) \leqslant N^\zeta$, where

$$\xi_N^{\pm} := N^{1/2}\left(\frac{N^{\pm}}{N} - \frac{1}{2}\right).$$

## Theorem [L., Poquet (2015)] :

Fix a constant $\tau_f$ and a phase $\psi_0 \in \mathbb{S}$ and an *admissible* sequence $(\omega_i)_{i \geqslant 1}$.
If for all $\varepsilon > 0$

$$\lim_{N\to\infty} \mathbf{P}\left(\|\mu_{N,0} - q_{\psi_0}\|_{-1} \leq \varepsilon\right) = 1,$$

then

$$\lim_{N\to\infty} \mathbf{P}\left(\sup_{\tau\in[0,\tau_f]} \left\|\mu_{N,N^{1/2}\tau} - q_{\psi_0+b(\xi_N)\tau}\right\|_{-1} \leq \varepsilon\right) = 1,$$

where $b$ is linear and satisfies for all $\xi$ such that $\xi^+ + \xi^- = 0$

$$b(\xi) = 2\xi^+\omega_0 + O(\omega_0^2), \text{ as } \omega_0 \to 0.$$

We want to quantify how close the empirical measure of the system $\mu_N$ is close to the manifold M. Introduce

$$\nu_{N,t} := \mu_{N,t} - q_\psi,$$

where $q_\psi$ is any element of $M$. Using Ito formula and introducing the semi-group $e^{tL_{q_\psi}}$, one can write a mild formulation in $H^{-1}$ of $\nu_N$ :

$$\nu_{N,t} = e^{tL_{q_\psi}}\nu_{N,0} + \int_0^t e^{(t-s)L_{q_\psi}} \left(D_{q,N} + R_{q,N}(\nu_{N,s})\right) \mathrm{d}s + Z_{N,t},$$

where

- $D_{q,N} = \partial_\theta \left(q_\psi \left\{ \left(\frac{N^+}{N} - \frac{1}{2}\right)(K \sin * q_\psi^+) + \left(\frac{N^-}{N} - \frac{1}{2}\right)(K \sin * q_\psi^-) \right\}\right)$ is the drift part induced by the asymmetry of the disorder,
- $R_{q,N}(\nu_N)$ is a quadratic nonlinearity,
- $Z_{N,t}$ is the noise part.

$$\nu_{N,t} = e^{tL_{q_\psi}} \nu_{N,0} + \int_0^t e^{(t-s)L_{q_\psi}} \left(D_{q,N} + R_{q,N}(\nu_{N,s})\right) \mathrm{d}s + Z_{N,t}.$$

We have a competition between two effects :

- the random influence of the drift $D_N$ and noise $Z_N$ that moves away $\mu_N$ from $M$
- the deterministic semigroup $e^{tL_q}$ that projects back $\mu_N$ along the normal direction $N_q$,
- the dynamics of $\mu_N$ is essentially along the neutral direction $T_q$ : we obtain a traveling wave whose speed depends only on the drift $D_N$
- Since $D_N$ is a linear functional of $\left(\frac{N^+}{N} - \frac{1}{2}\right)$, it is of order $\approx \frac{1}{\sqrt{N}}$, so one has to wait a time of order $\sqrt{N}$ in order to see this traveling wave.

# Control of the noise

The noise term is, for all test fonction $h$

$$Z_{N,t}(h) = \sum_{\sigma=\pm} \frac{1}{2N^\sigma} \sum_{j=1}^{N^\sigma} \int_0^t \partial_\theta \left[ \left( e^{(t-s)L_\psi^*} h \right)^\sigma \right] (\theta_j^\sigma(s)) \, \mathrm{d}B_j(s) \, ,$$

## Proposition

*For all* $\varepsilon > 0$ *and* $m > 0$*, there exists* $C_{m,\varepsilon} > 0$ *such that for all* $0 \leqslant s < t \leqslant T$*,*

$$\mathbf{E} \| Z_{N,t} - Z_{N,s} \|_{-1}^{2m} \leqslant \frac{C_{m,\varepsilon}}{N^m} \left( (t-s)^{m(1/2 - 2\varepsilon)} + (t-s)^m \right) . \qquad (1)$$

This requires to know that the semigroup $e^{tL_q}$ is somehow regularizing : one has fractional estimates of the type

$$\left\| e^{tL_q^*} u \right\|_{1+2\beta} \leqslant C \left( 1 + \frac{e^{-\gamma t}}{t^\beta} \right) \|u\|_1 \, , \qquad (2)$$

The procedure is to look at the dynamics of $\nu_N$ on $[0, \sqrt{N}T]$, discretized on subintervals $[nT, (n+1)T]$, $n = 0, \ldots, \lfloor \sqrt{N} \rfloor$.

Using the semi-martingale decomposition and the fact that the disorder is admissible, prove recursively that, if $d(\mu_{N,nT}, M) = O(N^{-1/2+2\zeta})$ for some $n$, then on the subinterval $[nT, (n+1)T]$, with high probability as $N \to \infty$,

- The drift $D_{N,t}$ and the noise $Z_{N,t}$ are of order $O(N^{-1/2+\zeta})$
- The nonlinearity $R_N$ is of order $O(N^{-1+4\zeta})$,
- $\nu_{N,t}$ is of order $O(N^{-1/2+2\zeta})$
- one can define recursively $q_{\psi_n}$ as the projection of $\mu_{N,nT}$ on $M$

$$q_{\psi_n} = P_{\psi_n}(\mu_{N,nT}).$$

The process $\nu_{n,t} := \mu_{N,nt} - q_{\psi_n}$, $t \in [0, T]$ is recursively well-defined and one has

## Proposition (a priori bound on $\nu_N$)

*With probability going to* $1$ *as* $N \to \infty$*,*

$$\sup_{1 \leqslant n \leqslant \lfloor \sqrt{N} \rfloor} \sup_{t \in [0,T]} \left\| \nu_{n,t} \right\|_{-1} \leqslant CN^{-1/2+2\zeta}.$$

The overall drift of the phase of $\mu_N$ induced by the dynamics on the interval $[0, \sqrt{N}T]$ is given by the sum of the small drifts induced by $D_N$ that is

$$T \sum_{n=1}^{\lfloor \sqrt{N} \rfloor} P_{\psi_{n-1}}(D_{\psi_{n-1}}) + O(N^{-1/4+2\zeta}),$$

with high probability as $N \to \infty$.

By rotation invariance

$P_{\psi_{n-1}}(D_{\psi_{n-1}}) = N^{-1/2} P_{\psi_{n-1}}(K\partial_\theta(\xi_N \cdot (\sin * q_{\psi_{n-1}})q_{\psi_{n-1}})$ does not depend on $\psi_{n-1}$ so that the whole drift is given by

$$b(\xi) := K P_0 \partial_\theta(\xi \cdot (\sin * q)q)$$

- Mean-field disordered system with rotational invariance,
- In the limit as $N \to \infty$, the population is equally balanced : no traveling wave in the thermodynamic limit.
- Quenched disorder induce random traveling waves for a finite population on a time scale of order $\sqrt{N}$.
- And what for time of order $N$ ?

Thank you for your attention !

E. Luçon and C. Poquet *Long time dynamics and disorder-induced traveling waves in the stochastic Kuramoto model*, arXiv :1505.00497.