Hydrodynamic limit for a collective motion model

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Plan of the talk

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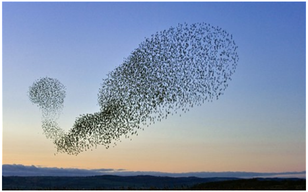
Collective motion

- Numerous examples of collective motions on most scales of the biological spectrum



- Collective motion arises without apparent leader or spatial constraints
- Most models are individual based models (IBM), and represented by *active matter*.

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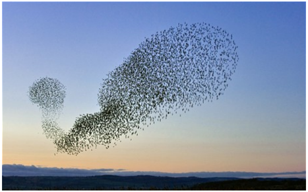

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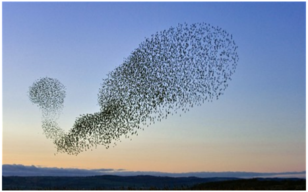

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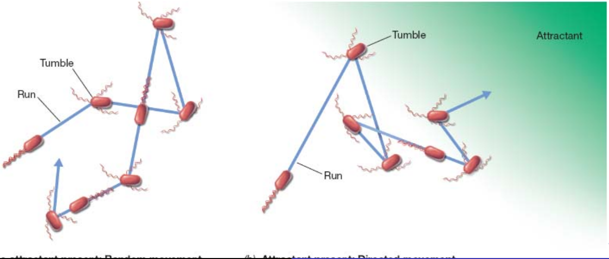
Active matter

Definition (Active matter)

Physical system composed of many individuals, maintained out of equilibrium by an *energy influx at the individual level* rather that on a global scale.

Example : "Run & Tumble" bacteria, E.Coli, Motility induced phase separation



Modelling of Collective dynamics : Alignment

Two types of phenomena can arise in collective dynamics
$\mapsto$ Alignment dynamics will create groups of particles moving together and adapting their speed

- Original model by Vicsek&al. (1995) presenting a phase transition
- Numerous simulations around related models, better understanding of the nature of the phase transition
- Exact work under an assumption of mean-field interaction : Bolley, Carillo, Degond

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Modelling of Collective dynamics : Motility induced phase separation

$\mapsto$ Motility induced phase separation (MIPS) can appear in active particle models

- The MIPS is due to the particles moving more slowly in crowded zones
- In models with strong alignment mechanisms, MIPS does not occur
- The convergence to equilibrium is prevented by the influx of energy at individual level

Goal : study a microscopic model which encompasses both physical phenomena (Alignement & MIPS) and obtain an exact scaling limit

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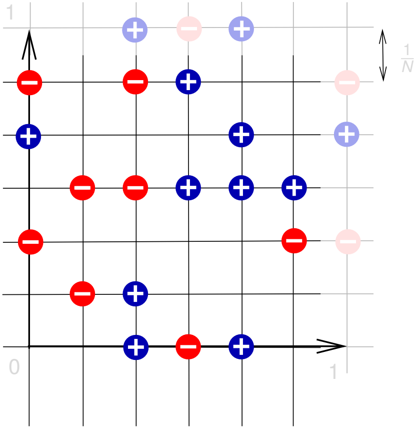
Plan of the talk

1. Collective motion
2. Model description
3. Hydrodynamic limit
   - Heuristic formulation of the macroscopic limit
   - Proper formulation of the Hydrodynamic limit
   - Key points in the proof
4. Conclusion
   - Continuous spin dynamic
   - Research perspectives

Description of the particle system

- Two types of particles ($+$ and $-$) evolve on the discrete torus $\mathbb{T}_N^2 = \left(\frac{1}{N}\mathbb{Z}/\mathbb{Z}\right)^2$
- For each site $x \in \mathbb{T}_N^2$, we define $\eta_x \in \{-1, 0, 1\}$ which describes whether the site is empty ($\eta_x = 0$),or occupied by a particle $\pm$ ($\eta_x = \pm 1$).



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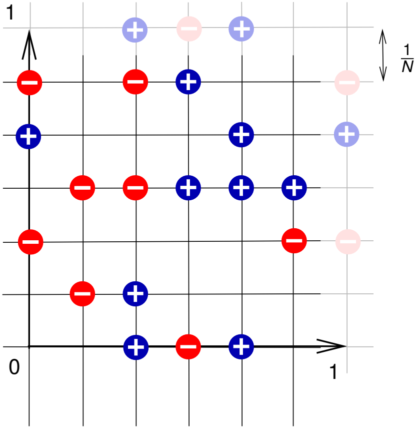


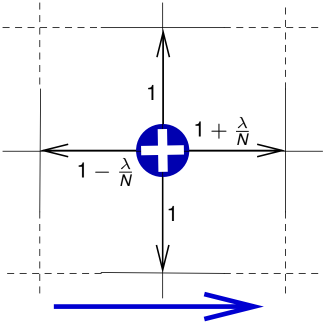
Initial configuration

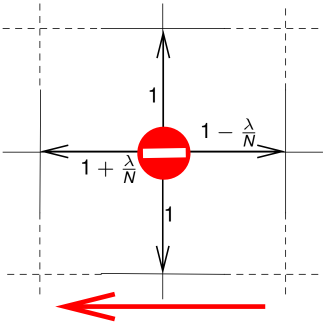
- The initial macroscopic density of $+$ particles (resp. $-$) is a fixed smooth functions ρ_0^+ (resp ρ_0^-), $(\mathbb{R}/\mathbb{Z})^2 \to [0, 1]$.
- We assume that for any $u \in (\mathbb{R}/\mathbb{Z})^2$,

$$\rho_0^+(u) + \rho_0^-(u) < 1.$$

- Each site x of the discrete torus is then occupied by a particle $+$ (resp. $-$) w.p. $\rho_0^+(x)$ (resp. $\rho_0^-(x)$), and we leave the site empty w.p. $1 - \rho_0^+(x) - \rho_0^-(x)$.

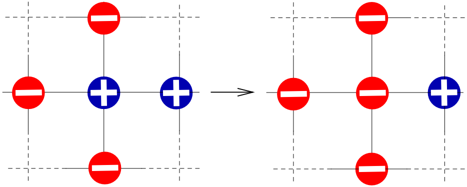
Exclusion dynamics





- We set a diffusive scaling on the exclusion process
- We tune the asymmetry via the parameter λ
- If the target site is occupied, the motion is canceled

Alignment dynamics



$\mapsto$ The alignment dynamics is tuned by the Ising-like inverse temperature β

- $\beta = 0$, the new spin is independent from the neighbors
- $\beta \to \infty$, the new spin is the most represented among occupied neighbors

Generator of the dynamic

The Markov generator of the process is given by

$$L_N = N^2 \mathcal{L}^S + N\mathcal{L}^{WA} + \mathcal{L}^G,$$

where

$$\mathcal{L}^S f(\eta) = \sum_{x\in\mathbb{T}_N} \sum_{|z|=1} |\eta_x| \left(1 - |\eta_{x+z}|\right) \left(f(\eta^{x,x+z}) - f(\eta)\right),$$

$$\mathcal{L}^{WA} f(\eta) = \sum_{x\in\mathbb{T}_N} \sum_{\delta=\pm 1} \lambda\delta\eta_x \left(1 - |\eta_{x+\delta e_1}|\right) \Big(f(\eta^{x,x+\delta e_1}) - f(\eta)\Big),$$

$$\mathcal{L}^G f(\eta) = \sum_{x\in\mathbb{T}_N} c(x,\eta)|\eta_x| \left(f(\eta^x) - f(\eta)\right).$$

Plan of the talk

Heuristic formulation of the macroscopic limit

Theorem

The macroscopic density of particles $+$, denoted $\rho^+(t,u)$, is a weak solution of the reaction-diffusion equation

$$\partial_t \rho^+ = \frac{1}{2}\nabla.\left[d_s(\rho)\nabla\rho^+ + D(\rho^+,\rho)\nabla\rho\right] + \lambda\nabla.\sigma(\rho^+,\rho) + \Gamma(\rho^+,\rho). \quad (1)$$

An equivalent equation is verified by the $-$ particle density ρ^-, and $\rho = \rho^+ + \rho^-$ is the total particle density.

$$\partial_t \rho^+ = \frac{1}{2}\nabla.\left[d_s(\rho)\nabla\rho^+ + D(\rho^+,\rho)\nabla\rho\right] + \lambda\nabla.\sigma(\rho^+,\rho) + \Gamma(\rho^+,\rho).$$

- D quantifies the diffusion due to heterogeneities of the total particle density
- d_s is the *self-diffusion* coefficient of a tracer particle in an homogeneous environment
- The coefficient σ is linked to the conductivity of the system
- Γ is the $+$ particles creation rate
- Both D and σ are explicit functions of ρ^+, ρ and $d_s(\rho)$.

Rigorous formulation of the macroscopic limit

Let us denote $\eta_x^\pm = \mathbb{1}_{\eta_x = \pm 1}$, we introduce the empirical measures of the process

$$\pi_N^+ = \sum_{x \in \mathbb{T}_N} \eta_x^+ \delta_x \qquad \text{and} \qquad \pi_N^- = \sum_{x \in \mathbb{T}_N} \eta_x^- \delta_x .$$

Let Q^N denote the law of the pair (π_N^+, π_N^-).

Theorem

The sequence $(Q^N)_{N \in \mathbb{N}}$ is weakly relatively compact. Furthermore, any of its limit points Q^* is concentrated on pair of measures (π^+, π^-) absolutely continuous w.r.t. the Lebesgue measure on the continuous torus, which densities ρ^+ and ρ^- are in H^1 and weak solutions of equation (1).

Irreducibility

- Due to the exclusion rule, and to the multiple types of particles, the process is not always irreducible on hyperplanes with fixed number of particles
- To circumvent this difficulty, one must prove that

$$\mathbb{E}\left(\frac{1}{N^2}\sum_{x\in\mathbb{T}_N}\mathbb{1}_{E_{p,x}}\right)\underset{p\to\infty}{\to}0, \qquad (2)$$

where $E_{p,x}$ is the event "there is less than one empty site in the box of size p around x."

Elements on the proof of (2)

The total density $\rho = \rho^+ + \rho^-$ is expected to satisfy

$$\partial_t \rho = \Delta\rho + \nabla.m(1-\rho),$$

where $m = \lambda(\rho^+ - \rho^-)$ denotes the local "magnetisation" of the system.

- The idea is to conduct an Analysis-based proof with purely microscopic method, using the PDE satisfied by ρ
- The proof relies on the a-priori control of the density and Gronwall's Lemma

Elements on the proof of (2)

Let $\phi(\rho) = 1/1 - \rho$

$$\partial_t \int_{\mathbb{T}} \phi(\rho_t) du = \int_{\mathbb{T}} \phi'(\rho_t) \left[\Delta \rho_t + \nabla.(m(1-\rho_t))\right] du$$

$$= \int_{\mathbb{T}} \phi''(\rho_t) \left[-(\nabla \rho_t)^2 - m(1-\rho_t)\nabla \rho_t\right] du$$

$$\leq \int_{\mathbb{T}} \phi''(\rho_t) \left[-(\nabla \rho_t)^2 + \frac{(\nabla \rho_t)^2}{2} + \lambda^2 (1-\rho_t)^2\right] du$$

$$\leq \int_{\mathbb{T}} \phi''(\rho_t) \lambda^2 (1-\rho_t)^2 du$$

$$= 2\lambda^2 \int_{\mathbb{T}} \phi(\rho_t) du$$

Elements on the proof of (2)

One can then apply Gronwall's Lemma to obtain that for any time t,

$$\int_{\mathbb{T}} \phi(\rho_t)du \leq e^{Ct} \int_{\mathbb{T}} \phi(\rho_0)du,$$

therefore if $\int_{\mathbb{T}} \phi(\rho)du$ is finite at $t = 0$, it should remain finite up to any time $t > 0$, and the total particle density should not reach 1.

$\mapsto$The challenge is to carry out this proof in a microscopic setup, without proving an hydrodynamic limit.

Out of equilibrium dynamics

- In order to obtain exact results (law of large numbers), one wants to work with the equilibrium measure, which is a product measure on the discrete torus
- The distortion of the measure due to the Glauber par and to the initial configuration are easily controlled
- The drift applied to the particles drives the system out of equilibrium
- We prove that exponential estimates needed in the *non-gradient* proof of the hydrodynamic limit are not hindered by the distortion of the measure of the process

Non-Gradient hydrodynamic limit

The core principle in the proof of the hydrodynamic limit is that

$$\partial_t \rho^+(x) = \partial_t \mathbb{E}(\eta_x^+) = \mathbb{E}(L_N \eta_x^+)$$

$$L_N \eta_x^+ = \sum_{i=1}^{2} (W^+_{x-\frac{e_i}{N},x} - W^+_{x,x+\frac{e_i}{N}}) + c_x(\eta),$$

where $W^+_{x,y}$ is the instantaneous current along the edge (x, y). It is composed of two parts, namely the symmetric and asymmetric currents :

$$W^+_{x,x+\frac{e_i}{N}} = N^2 W^{+,S}_{x,x+\frac{e_i}{N}} + \lambda N W^{+,A}_{x,x+\frac{e_i}{N}}.$$

- In the scaling limit $N \to \infty$, considering

$$L_N\left(\frac{1}{N^2}\sum_{x\in\mathbb{T}_N^2} G(x/N)\eta_x^+(t)\right)$$

the N factors can be balanced out by integration by parts into the discrete derivative of the smooth function G

- The system is *non gradient*, because the symmetric instantaneous current

$$W^{+,S}_{x,x+\frac{e_i}{N}} = \eta_x^+(1-|\eta_{x+\frac{e_i}{N}}|) - \eta^+_{x+\frac{e_i}{N}}(1-|\eta_x|)$$

is not a discrete gradient, and cannot therefore absorb the second factor N.

Plan of the talk

Continuous spin dynamic

Considering the two types of particles as having a preferred velocity angle 0 and π, We want to extend the proof of the hydrodynamic limit to a continuum of particle types, where each particle has an angle $\theta \in [0, 2\pi[$. Then, the macroscopic density ρ^θ of particles with angle θ is solution in the weak sense of

$$\partial_t \rho^\theta = \frac{1}{2}\nabla \left[d_s(\rho)\nabla \rho^\theta + D(\rho^\theta, \rho)\nabla \rho \right] + \lambda \nabla \sigma(\rho^\theta, \rho) + \Gamma^\theta$$

- $\rho = \int_\theta \rho^\theta d\theta$ is the total particle density
- Γ^θ depends on the type of alignment considered : continuous diffusion, jump process

Some research perspectives

- Change the nature of the alignment :
  Replace a metric interaction by a topological one
  (*k*-nearest neighbors)
- Does the model present a phase transition in terms of density/temperature/drift ?
  Problem : stationnary measure of the process with drift and alignment ?

Thanks for your attention !