Quenched invariance principle for random walks on Poisson-Delaunay triangulations

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Introduction

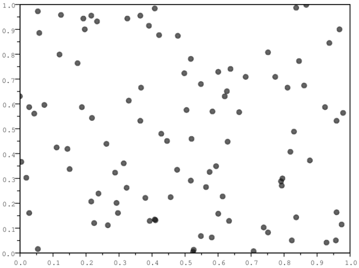
Result

Proof

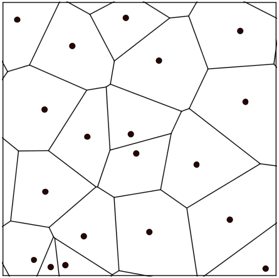
Poisson point processes

Homogeneous Poisson point process ξ with intensity λ (PPP)

1. $\forall k \geq 2$, $\forall A_1, \ldots, A_k \subset \mathbf{R}^d$ disjoint bounded Borel sets, the r.v. $\#(\xi \cap A_1), \ldots, \#(\xi \cap A_k)$ are independent
2. $\forall A \in \mathcal{B}_b(\mathbf{R}^d)$, $\#(\xi \cap A)$ is Poisson distributed with mean $\lambda \cdot \mathrm{Vol}(A)$



Voronoi tiling and Delaunay triangulation



- Voronoi cell with nucleus $x \in \xi$:

$$\mathrm{Vor}_\xi(x) = \left\{ y \in \mathbf{R}^d : |y - x|_2 \leq |y - x'|_2, \forall x' \in \xi \right\}$$

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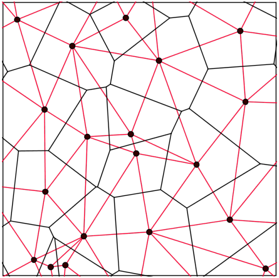

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- DT(ξ): Delaunay triangulation of ξ

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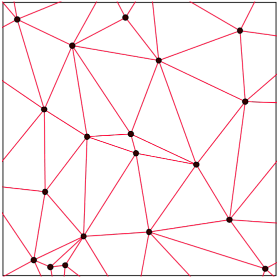

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Random walks in random environments in the literature

Models	Recurrence and transience	Invariance principles	
		annealed	*quenched*
Percolation cluster and random conductances in $\mathbf{Z}^d$	[Grimmett *et al.*; '93]	[De Masi *et al.*; '89]	[Berger, Biskup; '07], [Biskup, Prescott; '07], ...
Complete graph generated by point proc. in $\mathbf{R}^d$, transition probab. $\searrow$ with distance	[Caputo *et al.*; '09]	[Faggionato *et al.*; '06]	[Caputo *et al.*; '13],
Delaunay triangulation generated by PPP	([Addario-Berry, Sarkar; '05])		[Ferrari *et al.*; '12], ($d = 2$)

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Quenched invariance principle

- ξ distributed according to a PPP in $\mathbf{R}^d$, $\mathbf{d} \geq \mathbf{2}$
- $\left(X_n^\xi\right)_{n\in\mathbf{N}}$: simple nearest neighbor random walk on $\mathrm{DT}(\xi)$
- P_x^ξ: law of $\left(X_n^\xi\right)_{n\in\mathbf{N}}$ starting at x
- $B_\varepsilon^\xi(t) = \varepsilon\left(X^\xi_{\lfloor\varepsilon^{-2}t\rfloor} + \left(\varepsilon^{-2}t - \lfloor\varepsilon^{-2}t\rfloor\right)\left(X^\xi_{\lfloor\varepsilon^{-2}t\rfloor+1} - X^\xi_{\lfloor\varepsilon^{-2}t\rfloor}\right)\right), \quad t \geq 0$

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Theorem [R.; '15]

For all $T > 0$, for *a.e.* ξ, for all $x \in \xi$, the law of $\left(B_\varepsilon^\xi(t)\right)_{0\leq t\leq T}$ induced by P_x^ξ on $\mathcal{C}([0,T];\mathbf{R}^d)$ converges weakly, as $\varepsilon \to 0$, to the law of a Brownian motion $\left(B_t^\xi\right)_{0\leq t\leq T}$ starting at x with covariance matrix $\sigma^2\,\mathrm{I}_d$ where σ^2 is positive and does not depend on ξ.

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Martingale decomposition

For *a.e.* ξ, for $x \in \xi$, we want to write:

$$X_n^\xi = M_n^\xi + R_n^\xi$$

with

$\left(M_n^\xi\right)_{n\in\mathbf{N}}$: P_x^ξ-martingale
$\hookrightarrow$ converges to a BM by Lindeberg-Feller functional CLT

and

$\left(R_n^\xi\right)_{n\in\mathbf{N}}$: corrector

$\hookrightarrow$ negligible at the diffusive scale: $\lim_{n\to\infty} \frac{R_n^\xi}{n} = 0$ *a.s.*

Construction of the martingale (1/3)

Let μ be the measure on $\mathcal{N}_0 \times \mathbf{R}^d$ defined by:

$$\int f \, \mathrm{d}\,\mu = \int_{\mathcal{N}_0} \sum_{x \underset{\xi^0}{\sim} 0} (f(x) - f(0)) \, \mathcal{P}_0(\mathrm{d}\,\xi^0),$$

where $\mathcal{P}_0$ denotes the Palm measure associated to the PPP.

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Weil decomposition of $\mathrm{L}^2(\mu)$

$$\mathrm{L}^2(\mu) = \mathrm{L}^2_{\mathrm{pot}}(\mu) \overset{\perp}{\oplus} \mathrm{L}^2_{\mathrm{sol}}(\mu)$$

with

$\mathrm{L}^2_{\mathrm{pot}}(\mu)$: closure of the space of gradients of bounded meas. functions

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Consider the projection

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Note that it is in $\mathrm{L}^2(\mu)$ since

$$\begin{aligned}\int |p|^2 \,\mathrm{d}\,\mu &= \int_{\mathcal{N}_0} \sum_{x \underset{\xi^0}{\sim} 0} |x|^2 \mathcal{P}_0(\mathrm{d}\,\xi^0) \\ &\leq \int_{\mathcal{N}_0} \deg_{\xi^0}(0) \max_{x \underset{\xi^0}{\sim} 0} |x|^2 \mathcal{P}_0(\mathrm{d}\,\xi^0) \\ &\leq \left(\int_{\mathcal{N}_0} \deg_{\xi^0}(0)^2 \mathcal{P}_0(\mathrm{d}\,\xi^0)\right)^{\frac{1}{2}} \left(\int_{\mathcal{N}_0} \max_{x \underset{\xi^0}{\sim} 0} |x|^4 \mathcal{P}_0(\mathrm{d}\,\xi^0)\right)^{\frac{1}{2}} < \infty.\end{aligned}$$

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So, we can write

$$p = \underset{\in \mathrm{L}^2_{\mathrm{pot}}(\mu)}{\chi} + \underset{\in \mathrm{L}^2_{\mathrm{sol}}(\mu)}{\varphi}.$$

Construction of the martingale (3/3)

Since $\varphi \in \mathrm{L}^2_{\mathrm{sol}}(\mu)$ is antisymmetric

$$\sum_{x \underset{\xi^0}{\sim} 0} \varphi\left(\xi^0, x\right) = 0, \qquad \text{for } \mathcal{P}_0\text{-}a.e.\ \xi^0,$$

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and actually

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Thus,

$$M_n^\xi = \sum_{i=0}^{n-1} \varphi\left(\tau_{X_i^\xi} \xi, X_{i+1}^\xi - X_i^\xi\right) = \varphi\left(\tau_{X_0^\xi} \xi, X_n^\xi - X_0^\xi\right)$$

is a P_x^ξ-martingale.

The corrector

$$R_n^\xi = X_n^\xi - M_n^\xi = \chi\left(\tau_{X_0^\xi}\xi, X_n^\xi - X_0^\xi\right)$$

It remains to prove that

$$\lim_{n\to+\infty} \max_{y\in\xi\cap[-n,n]^d} \frac{|\chi(\tau_x\xi, y-x)|}{n} = 0 \ a.s..$$

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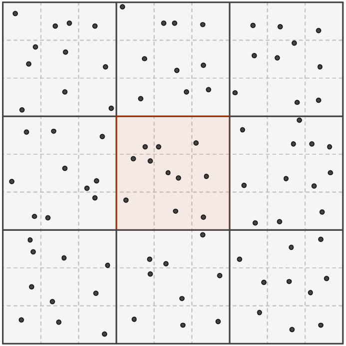
By the maximum principle, it suffices to show that

$$\lim_{n\to+\infty} \max_{y\in\mathcal{G}_\infty(\xi)\cap[-n,n]^d} \frac{|\chi(\tau_x\xi, y-x)|}{n} = 0 \; a.s.$$

where $\mathcal{G}_\infty(\xi)$ is an infinite connected subgraph of $\mathrm{DT}(\xi)$ such that each connected component of $\mathrm{DT}(\xi)\setminus\mathcal{G}_\infty(\xi)$ is finite.

Construction of $\mathcal{G}_\infty(\xi)$ (1/3)



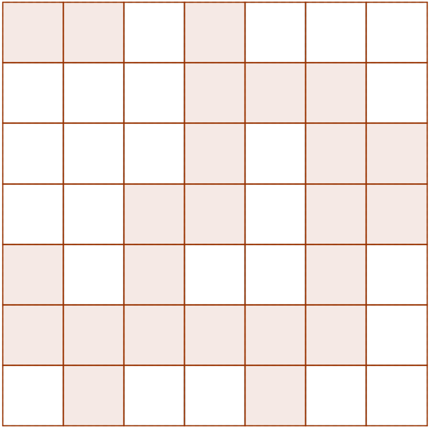
We part $\mathbf{R}^d$ into boxes $B_{\mathbf{z}}$ of side K, $\mathbf{z} \in \mathbf{Z}^d$, and subdivise each box into sub-boxes of side $\alpha_d K$.

We say that $B_{\mathbf{z}}$ is good if:

- each sub-box of side $\alpha_d K$ included in $\overline{B_{\mathbf{z}}} = \bigcup_{|\mathbf{z}'-\mathbf{z}|\leq 1} B_{\mathbf{z}'}$ contains at least a point of ξ,
- $\#\left(\xi \cap \overline{B_{\mathbf{z}}}\right) \leq D$.

If K and D are well chosen, the process of the good boxes stochastically dominates an indep. percolation process with parameter $p \in (1 - p_c(\mathbf{Z}^d), 1)$.

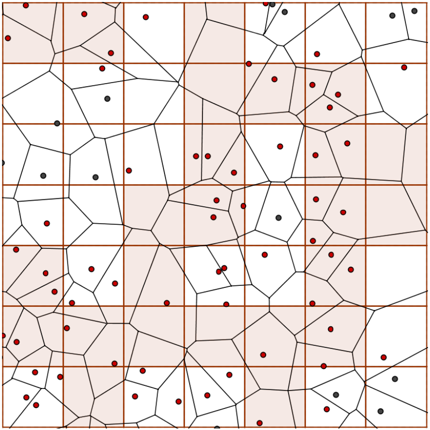
Construction of $\mathcal{G}_\infty(\xi)$ (2/3)



$\mathbb{G}_\infty =$ 'the infinite cluster of percolation'

$\mathbb{G}_L =$ 'the maximal connected component of $\mathbb{G}_\infty \cap [-L, L]^d$'

Construction of $\mathcal{G}_\infty(\xi)$ (3/3)



$$\mathcal{G}_\infty(\xi) = \{x \in \xi : \exists \mathbf{z} \in \mathbb{G}_\infty \text{ s.t. } \mathrm{Vor}_\xi(x) \cap B_{\mathbf{z}} \neq \emptyset\}$$

$$\mathcal{G}_L(\xi) = \{x \in \xi : \exists \mathbf{z} \in \mathbb{G}_L \text{ s.t. } \mathrm{Vor}_\xi(x) \cap B_{\mathbf{z}} \neq \emptyset\}$$

Sublinearity of the corrector in $\mathcal{G}_\infty(\xi)$ 'à la [Biskup, Prescott, '07]' (1/3)

Sublinearity on average:

$$\forall \varepsilon > 0, \lim_{n \to +\infty} \frac{1}{n^d} \sum_{y \in \mathcal{G}_\infty(\xi) \cap [-n,n]^d} \mathbf{1}_{|\chi(\tau_x \xi, y-x)| \geq \varepsilon n} = 0$$

- ergodicity arguments
- directional sublinearity
- extension dimension by dimension

Sublinearity of the corrector in $\mathcal{G}_\infty(\xi)$ 'à la [Biskup, Prescott, '07]' (2/3)

Polynomial growth:

$$\exists \theta > 0,\ \lim_{n\to\infty} \max_{y\in\mathcal{G}_\infty(\xi)\cap[-n,n]^d} \frac{|\chi(\tau_x\xi, y-x)|}{n^\theta} = 0$$

- analytic properties of χ

Sublinearity of the corrector in $\mathcal{G}_\infty(\xi)$ 'à la [Biskup, Prescott, '07]' (3/3)

Diffusive bounds:

Define $T_1 = \inf\{j \geq 1 : X_j^\xi \in \mathcal{G}_\infty(\xi)\}$.

The random walk $(Y_t^\xi)_{t\geq 0}$ with generator

$$\left(\mathcal{L}^\xi f\right)(y) = \sum_{y' \in \mathcal{G}_\infty(\xi)} P_y^\xi\left[X_{T_1}^\xi = y'\right]\left(f(y') - f(y)\right)$$

satisfies

$$\sup_{n\geq 1} \max_{y\in\mathcal{G}_\infty(\xi)\cap[-n,n]^d} \sup_{t\geq n} \max\left(t^{-\frac{1}{2}} E_y^\xi\left[|Y_t^\xi - y|\right], t^{-\frac{d}{2}} P_y^\xi\left[Y_t^\xi = y\right]\right) < +\infty \text{ p.s.}$$

- distance comparison
- **isoperimetric inequalities**
- heat kernel estimates for $(Y_t^\xi)_{t\geq 0}$ (see [Morris, Peres; '05])

Isoperimetric inequality in $\mathcal{G}_L(\xi)$ (1/5)

For $A \subset \mathcal{G}_L(\xi)$, define

$$I_A^{L,\xi} = \frac{\sum_{x \in A} \sum_{y \in A^c} \mathbf{1}_{x \sim y \text{ in } \mathrm{DT}(\xi)}}{\deg_L(A)}$$

where $A^c = \mathcal{G}_L(\xi) \setminus A$ and $\deg_L(A) = \sum_{x \in A} \deg_L(x)$.

Claim

There exists $c > 0$ such that *a.s.* for L large enough

$$I_A^{L,\xi} \geq c \min \left(\frac{1}{\deg_L(A)^{\frac{1}{d}}}, \frac{1}{\log(L)^{\frac{d}{d-1}}} \right)$$

for every $A \subset \mathcal{G}_L(\xi)$ with $\deg_L(A) \leq \frac{1}{2} \deg_L(\mathcal{G}_L(\xi))$.

Isoperimetric inequality in $\mathcal{G}_L(\xi)$ (2/5)

For $A \subset \mathcal{G}_L(\xi)$, define

$$\mathbb{L}(A) = \{\mathbf{z} \in \mathbb{G}_L : \exists x \in A \text{ s.t. } \mathrm{Vor}_\xi(x) \cap B_{\mathbf{z}} \neq \emptyset\} .$$

Note that

$$\frac{\#\mathbb{L}(A)}{2^d} \leq \deg_L(A) \leq \underbrace{\#(A)}_{\leq D\#\mathbb{L}(A)} \underbrace{\max_{x \in A} \deg_L(x)}_{\leq D} \leq D^2 \#\mathbb{L}(A). \tag{1}$$

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We distinguish the cases whether or not $\#\mathbb{L}(A) > \left(1 - \frac{1}{2^{d+2}D^2}\right) \#\mathbb{G}_L$.

Isoperimetric inequality in $\mathcal{G}_L(\xi)$ (3/5): case $\#\mathbb{L}(A) > \left(1 - \frac{1}{2^{d+2}D^2}\right) \#\mathbb{G}_L$

Since $\deg_L(A) \leq \frac{1}{2}\deg_L(\mathcal{G}_L(\xi))$, we have

$$\#\mathbb{L}(A^c) \geq \frac{\#\mathbb{G}_L}{2^{d+1}D^2} \quad \text{and} \quad \#\left(\mathbb{L}(A) \cap \mathbb{L}(A^c)\right) \geq \frac{\#\mathbb{G}_L}{2^{d+1}D^2}.$$

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If $\mathbf{z} \in \mathbb{L}(A) \cap \mathbb{L}(A^c)$, there exists an edge between a point of A and a point of A^c contained in $\overline{B_{\mathbf{z}}} = \bigcup_{|\mathbf{z}'-\mathbf{z}|\leq 1} B_{\mathbf{z}'}$.

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This implies that

$$\sum_{x \in A} \sum_{y \in A^c} \mathbf{1}_{x \sim y} \geq \frac{\#\left(\mathbb{L}(A) \cap \mathbb{L}(A^c)\right)}{3^d} \geq \frac{\#\mathbb{G}_L}{4 \times 6^d \times D^2}$$

so that

$$I_A^{L,\xi} \geq \frac{1}{4 \times 6^d \times D^4}.$$

Isoperimetric inequality in $\mathcal{G}_L(\xi)$ (4/5): case $\#\mathbb{L}(A) \leq \left(1 - \frac{1}{2^{d+2}D^2}\right) \#\mathbb{G}_L$

If $\mathbf{z} \in \mathbb{L}(A)$ and $\mathbf{z}' \in \mathbb{G}_L \setminus \mathbb{L}(A)$ are neighbors, there exists an edge between a point of A and a point of A^c contained in $\overline{B_{\mathbf{z}}} \cup \overline{B'_{\mathbf{z}}}$.

It follows that

$$\sum_{x \in A} \sum_{y \in A^c} \mathbf{1}_{x \sim y} \geq \delta \max\left(\#\left(\partial \mathbb{L}(A)\right), \#\left(\partial\left(\mathbb{G}_L \setminus \mathbb{L}(A)\right)\right)\right).$$

Isoperimetric inequality in $\mathcal{G}_L(\xi)$ (4/5): case $\#\mathbb{L}(A) \leq \left(1 - \frac{1}{2^{d+2}D^2}\right)\#\mathbb{G}_L$

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Besides,

$$\#\mathbb{L}(A) \leq \left(1 - \frac{1}{2^{d+2}D^2}\right)\left(\#\mathbb{L}(A) + \#\left(\mathbb{G}_L \setminus \mathbb{L}(A)\right)\right).$$

Hence,

$$deg_L(A) \leq D^2\#\mathbb{L}(A) \leq D^2(2^{d+2}D^2 - 1)\#\left(\mathbb{G}_L \setminus \mathbb{L}(A)\right)$$

and

$$I_A^{L,\xi} \geq \frac{\delta}{D^2(2^{d+2}D^2-1)}\frac{\#\left(\partial\mathbb{A}\right)}{\#\mathbb{A}}$$

for $\mathbb{A} = \mathbb{L}(A)$ or $\mathbb{G}_L \setminus \mathbb{L}(A)$.

Isoperimetric inequality in $\mathcal{G}_L(\xi)$ (5/5): case $\#\mathbb{L}(A) \leq \left(1 - \frac{1}{2^{d+2}D^2}\right) \#\mathbb{G}_L$

By applying

Isoperimetric inequality in $\mathbb{G}_L$ (see *e.g.* [Caputo, Faggionato; '07])

There exists $\kappa > 0$ such that almost surely for L large enough, for $\mathbb{A} \subset \mathbb{G}_{(L)}$ with $0 < \#(\mathbb{A}) \leq \frac{1}{2}\#(\mathbb{G}_{(L)})$

$$\frac{\#(\partial\mathbb{A})}{\#(\mathbb{A})} \geq \kappa \min\left\{\frac{1}{\#(\mathbb{A})^{\frac{1}{d}}}, \frac{1}{\log(L)^{\frac{d}{d-1}}}\right\}.$$

and then (1), one finally obtains that

$$I_A^{L,\xi} \geq \frac{\kappa\delta}{D^2(2^{d+2}D^2-1)} \min\left(\frac{1}{\#(\mathbb{A})^{\frac{1}{d}}}, \frac{1}{\log(L)^{\frac{d}{d-1}}}\right)$$
$$\geq \frac{\kappa\delta}{2D^2(2^{d+2}D^2-1)} \min\left(\frac{1}{\deg_L(A)^{\frac{1}{d}}}, \frac{1}{\log(L)^{\frac{d}{d-1}}}\right).$$

Thank you!