# Hydrodynamic spectrum of one-dimensional bulk-driven particle gases

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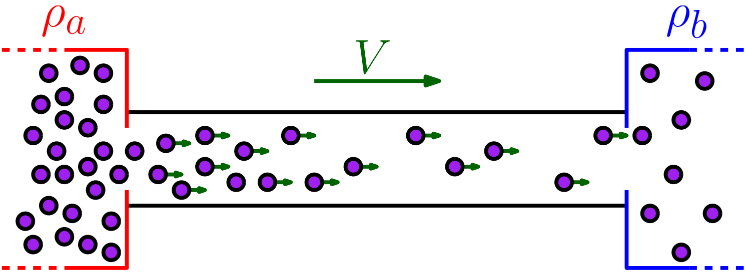
# Introduction

Simple picture: one-dimensional conduit, two reservoirs $\rho_a$ and $\rho_b$, a driving field $V$ in the bulk, and interactions between the particles.



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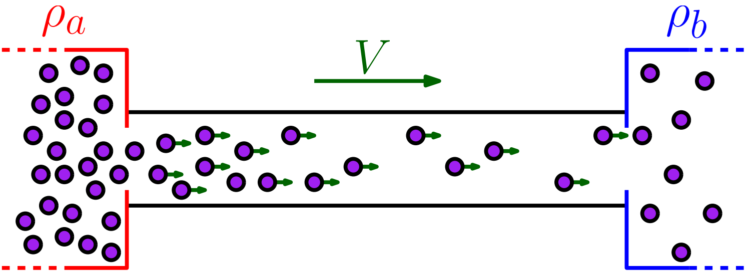



Field and/or reservoir imbalance $\Rightarrow$ macroscopic current of particles (related to microscopic production of entropy)

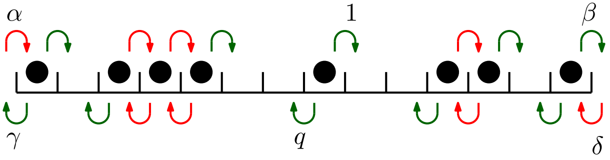
# Layout

- Introduction

- I – The open ASEP: definition and steady state
- II – Macroscopic Fluctuation Theory and current fluctuations
- III – Scope and limitations

- Conclusion

# I - The open Asymmetric Simple Exclusion Process




The open Asymmetric Simple Exclusion Process (ASEP):

- one-dimensional lattice of size $L$
- entry at the left with rate $\alpha$ and at the right with rate $\delta$
- exit at the right with rate $\beta$ and at the left with rate $\gamma$
- jumps in the bulk with rate $p = 1$ to the right and $q < 1$ to the left (if the target site is free)

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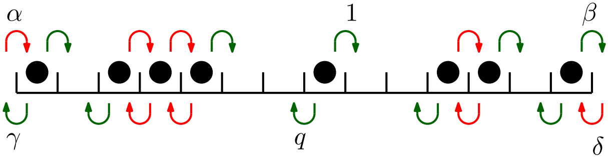


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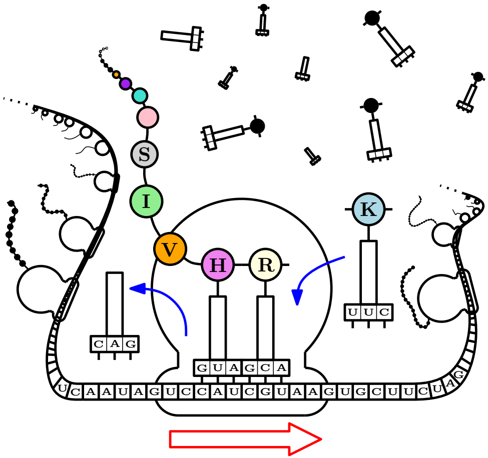
Totally Asymmetric case (TASEP): $q = \gamma = \delta = 0$

# I - The open Asymmetric Simple Exclusion Process

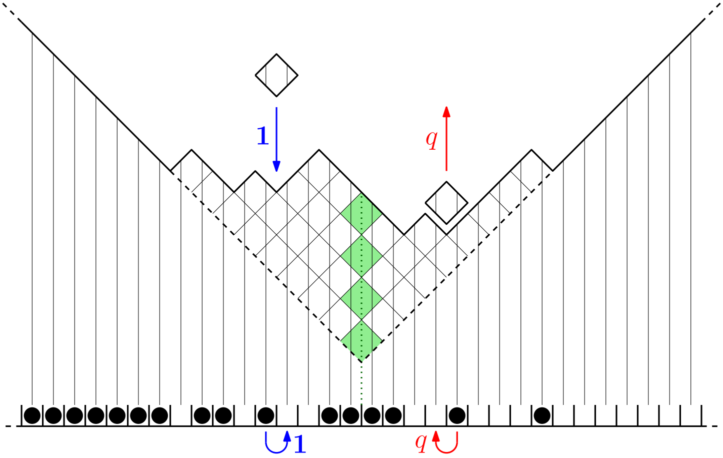
First invented to describe biological transport.




[C. T. MacDonald, J. H. Gibbs, A. C. Pipkin, **Biopolymers**, 1968]

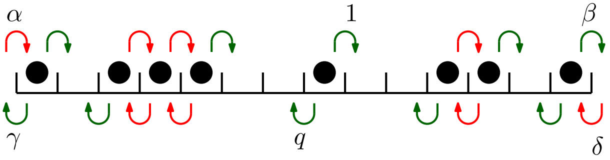
# I - The open Asymmetric Simple Exclusion Process

Can be related to other statistical models, such as surface growth.




[M. Kardar, G. Parisi, Y.-C. Zhang, **P. R. L.**, 1986]

# I - Motivation

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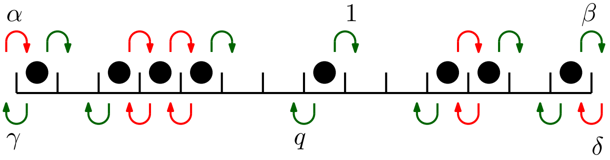


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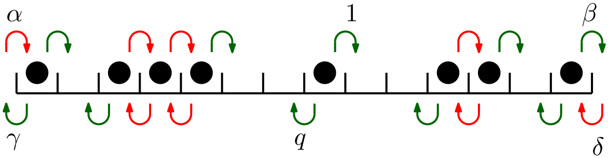


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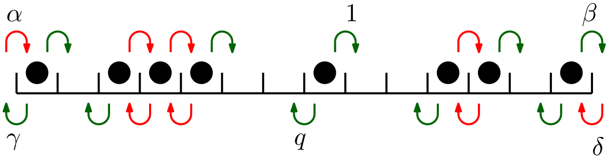


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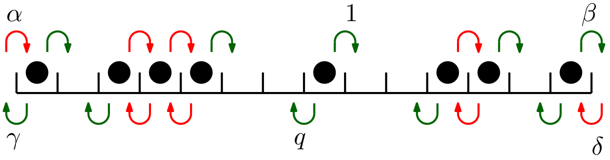


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What in the open ASEP ?

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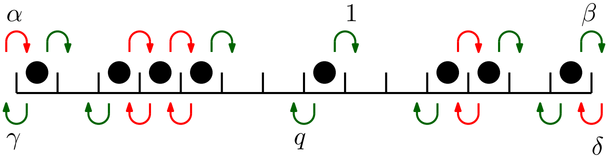


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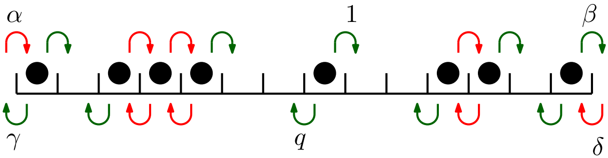
What in the open ASEP ? The macroscopic current of particles, related to the entropy production (irreversibility).

# I - Master equation




The probability vector $|P_t\rangle$ which contains the probabilities of observing a configuration $\mathcal{C}$ at time $t$ obeys the master equation

$$\frac{d}{dt}|P_t\rangle = M|P_t\rangle$$

with $M$ being the sum of local matrices $M_i$ (one for each bond $0 \leq i \leq L$) (in bases $\{0,1\}$ and $\{00,01,10,11\}$)

$$M_0 = \begin{bmatrix} -\alpha & \gamma \\ \alpha & -\gamma \end{bmatrix} \;,\; M_i = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -q & 1 & 0 \\ 0 & q & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \;,\; M_L = \begin{bmatrix} -\delta & \beta \\ \delta & -\beta \end{bmatrix}$$

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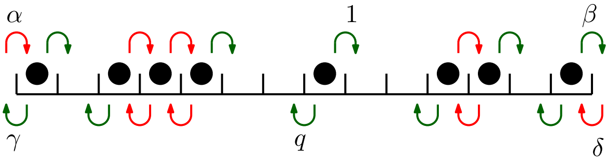


The probability vector $|P_t\rangle$ which contains the probabilities of observing a configuration $\mathcal{C}$ at time $t$ obeys the master equation

$$|P_t\rangle \to |P^\star\rangle$$

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Exact solution: [B. Derrida, M. R. Evans, V. Hakim, V. Pasquier, 1993]

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Mean field equation: $\boxed{J = (1-q)\rho(1-\rho) - \frac{1+q}{2L}\nabla\rho}$ with $\rho_a$, $\rho_b$.

For a given $J = (1-q)\rho_c(1-\rho_c)$, we plot all the possible profiles $\rho(x)$:

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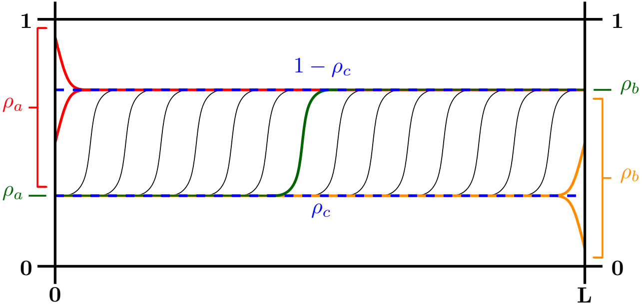
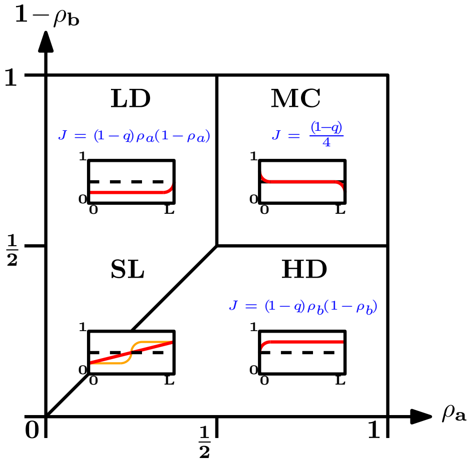



which gives us constraints on $\rho_a$ and $\rho_b$.

# I - Phase diagram of the steady state

Phase diagram of the system (with respect to $\rho_a(\alpha, \gamma, q)$ and $\rho_b(\beta, \delta, q)$):




In each case: $J = (1-q)\rho_c(1-\rho_c)$.

# II - Macroscopic fluctuations

**Macroscopic fluctuation theory**
**[Bertini, De Sole, Gabrielli, Jona-Lasinio, Landim; J. Stat. Phys, 2002]**

For a diffusive particle gas, the large deviations $g(j,\rho)$ are locally Gaussian around the deterministic equation, i.e. :

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We get, for the total current $j$ (through all bonds):

$$g(j,\rho) = \int_0^1 \frac{\left[j - \nu\rho(1-\rho) + \frac{1+q}{2}\nabla\rho\right]^2}{2\rho(1-\rho)} dx$$

Minimising over $\rho \to g(j)$ and optimal profile.

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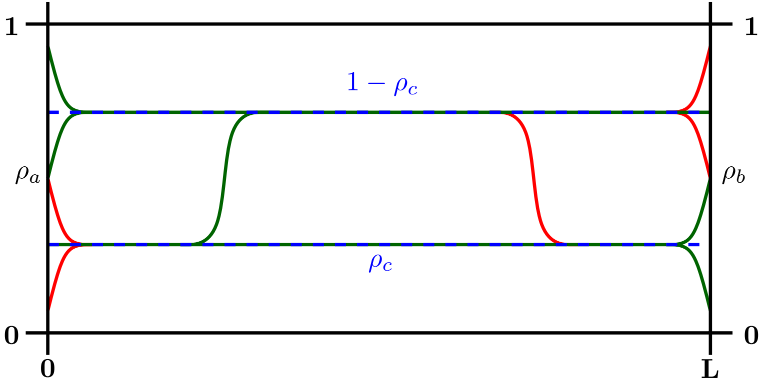
- [T. Bodineau, B. Derrida, **J. Stat. Phys**, 2006] :

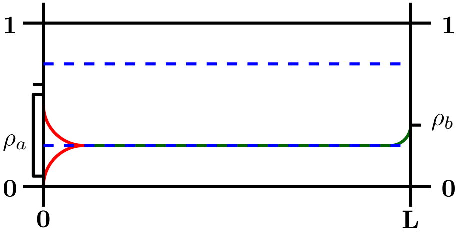
taking $\nu \sim L$ gives correct results for the TASEP.

# II - Macroscopic fluctuations

variations of $\rho = \pm$ steady state
for $j = \rho_c(1-\rho_c)$ (TASEP)



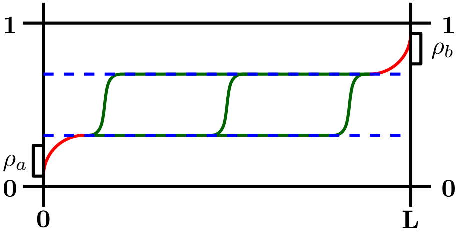
# II - LD/HD phases




$$g(j) = j \log\left(\frac{1-\rho_a}{\rho_a}\frac{\rho_c}{1-\rho_c}\right) + \rho_a - \rho_c$$

with $j = \rho_c(1-\rho_c)$

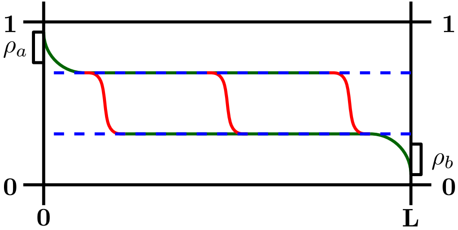
# II - Shock phase




$$g(j) = j \log\left(\frac{1-\rho_a}{\rho_a}\frac{\rho_b}{1-\rho_b}\frac{\rho_c^2}{(1-\rho_c)^2}\right) + \rho_a - \rho_b + 1 - 2\rho_c$$

$$\text{with } j = \rho_c(1-\rho_c)$$

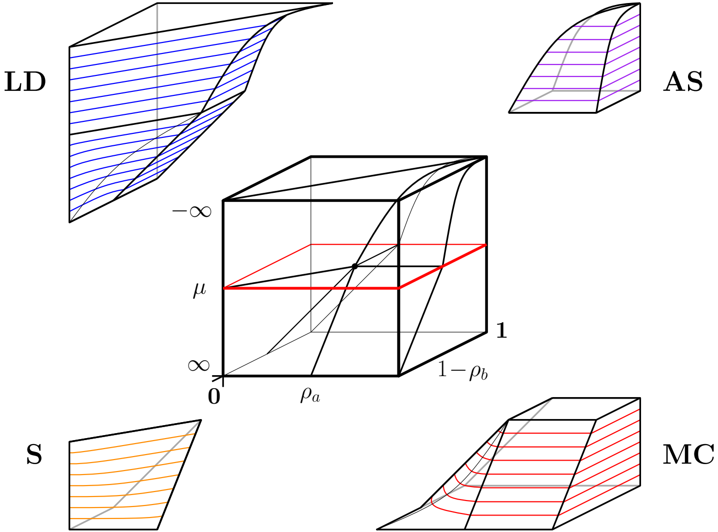
# II - Anti-shock phase




$$g(j) = 2j \log\left(\frac{\rho_c}{1-\rho_c}\right) + 1 - 2\rho_c$$

with $j = \rho_c(1-\rho_c)$

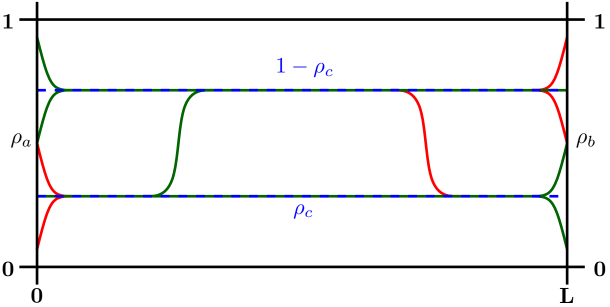
# II - Phase diagram



# III - Excited states from hydrodynamics

(Work in progress)

$$g(j,\rho)=\int_0^1 \frac{\left[j-\nu\rho(1-\rho)+\frac{1+q}{2}\nabla\rho\right]^2}{2\rho(1-\rho)}dx$$




We only looked at the most probable states. What about the others ?

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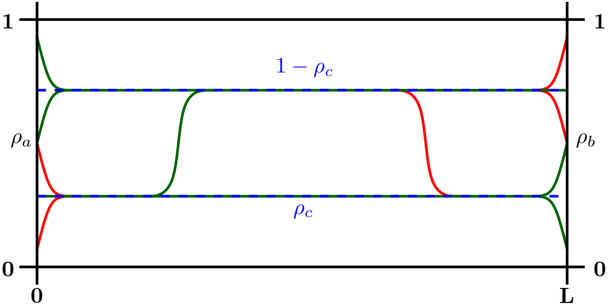


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We find that, for some, $\exists\ j, \frac{\mathrm{d}}{\mathrm{d}j} g(j) = 0 \rightarrow$ non-constrained states.

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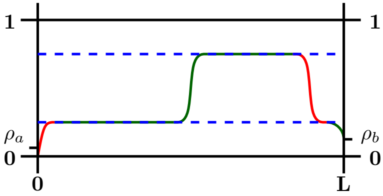


$$g(j) = j \log\left(\frac{1-\rho_a}{\rho_a}\left(\frac{\rho_c}{1-\rho_c}\right)^3\right) + 1 + \rho_a - 3\rho_c$$

$$\text{for } \rho_c = \left[1 + \left(\frac{1-\rho_a}{\rho_a}\right)^{1/3}\right]^{-1}$$

Exists if $\rho_b \leq 1 - \rho_c$.

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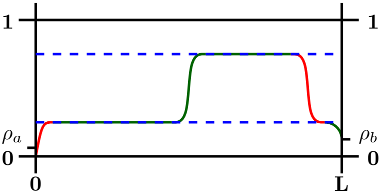


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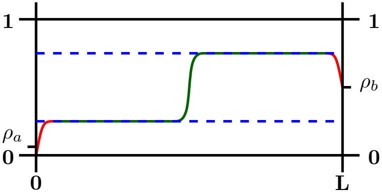


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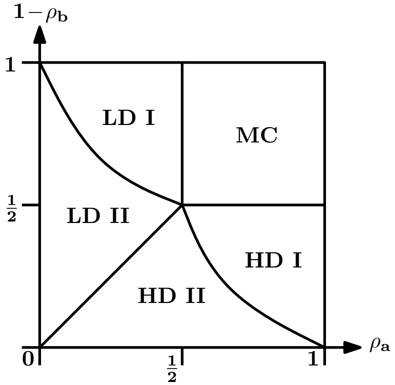
for $\rho_c = \left[1+\left(\frac{1-\rho_a}{\rho_a}\frac{\rho_b}{1-\rho_b}\right)^{1/2}\right]^{-1}$

Exists only if $\rho_b \geq \rho_c$.

# III - Excited states from hydrodynamics

$\rightarrow$ phase diagram for the gap:



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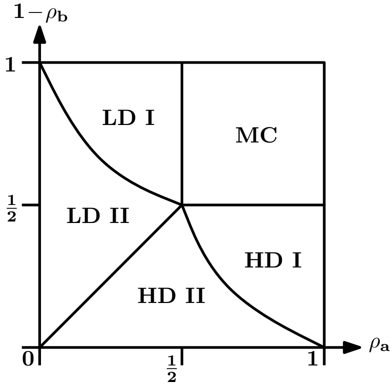
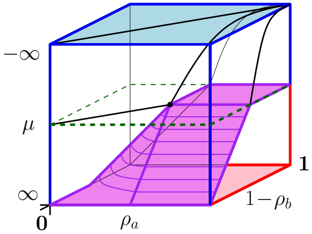



Values for the gap consistent with [F. Essler, J. de Gier, **J. Stat. Mech**, 2006] and [A. Proeme, R. Blythe, M. Evans, **J. Phys. A**, 2011] (numerics).

# III - Dynamical phase transition

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$$g(j) \sim L\ \varepsilon^2.$$

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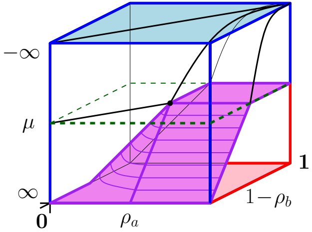


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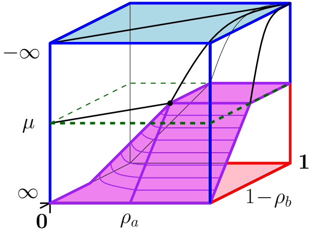
$$g(j) \sim L\ \varepsilon^{5/2}\ \frac{32\sqrt{3}}{5\pi}.$$

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$\rightarrow$ the structure of fluctuating states for $j > \frac{1}{4}$ is not hydrodynamic.

# III - Universal dynamical phase transition

(Work in progress)

$$M_0 = \begin{bmatrix} -\alpha & 0 \\ \alpha e^{\mu} & 0 \end{bmatrix} \ , \ M_i = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1e^{\mu} & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \ , \ M_L = \begin{bmatrix} 0 & \beta e^{\mu} \\ 0 & -\beta \end{bmatrix}$$

with $\mu \to \pm\infty$.

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- For $\mu \to -\infty$: main eigenvalue, and $g(j)$, independent of $L$.

$\to$ same transition, related to coarse-graining in the hydrodynamic limit.

# Conclusion - Open questions

- What is the driving term and the diffusion/noise in general ?

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- Is this generalisable to 2D systems ?

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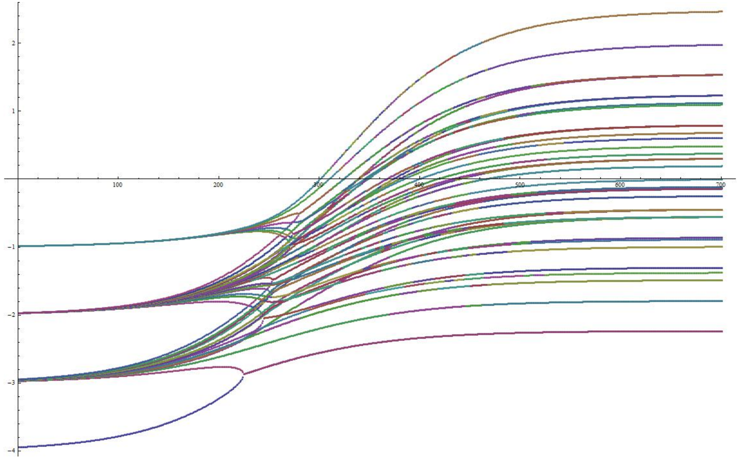
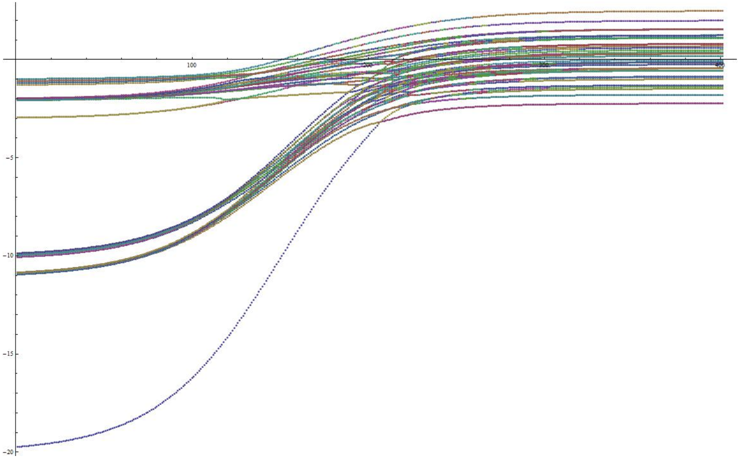
# Thank you !

TASEP.



TASEP with Ising interaction.



TASEP with disordered potential.