RANDOM POLYMERS

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§ INTRODUCTION

A polymer is a long chain of molecules, often referred to as monomers, with two characteristic properties:

(1) The chain is irregularly distributed in space.

(2) The chain interacts with itself and with its environment.

Examples of materials built out of polymers are: cellulose, plastic, rubber, soap.

A random polymer is a probabilistic model of a polymer incorporating (1) and (2). Many different models have been investigated. In this mini-course we review some of them.

The main type of question we address is:

What is the spatial distribution of the polymer as a function of its length and of its interaction parameters?

Random polymers represent a whole new area of probability theory, because they are random processes with a long-range interaction. Their behavior is very different from that of random processes with a short-range interaction, like Markovian models in space and/or time.

Mathematically, polymers are still rather ill understood. However, important progress has been made in the last 20 years. In addition, simulations have become a powerful guide.

In this mini-course we take a snapshot of several different models and describe the main theorems and conjectures for each of them.

LECTURE 1:

Basic polymer models

§ DOMB-JOYCE MODEL

Let $(S_i)_{i\in\mathbb{N}_0}$ be simple random walk on $\mathbb{Z}^d$, starting at the origin ($S_0 = 0$). Let P denote its law on path space. Fix a parameter $\beta \in [0,\infty]$ and for $n \in \mathbb{N}_0$ define a new law Q_n^β on the space of paths of length n by putting

$$\frac{dQ_n^\beta}{dP}(\cdot) = \frac{1}{Z_n^\beta}\, e^{-H_n^\beta(\cdot)},$$

where

$$H_n^\beta((S_i)_{i=0}^n) = \beta \sum_{0\le i<j\le n} 1\{S_i = S_j\}$$

is β times the number of self-intersections after n steps and Z_n^β is the normalizing constant.

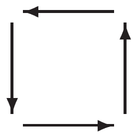
Under the law Q_n^β each self-intersection is penalized by a factor $e^{-\beta}$. The parameter β is the strength of self-repellence.

Two boundary cases:

srw: $Q_n^0(\cdot) = P(\cdot)$,

saw: $Q_n^\infty(\cdot) = P(\cdot \mid S_i \neq S_j \; \forall \, 0 \leq i < j \leq n)$.

For $\beta \in (0, \infty)$, self-intersections are not forbidden but are suppressed. Thus, paths that are "spread out" are favored over paths that are "rolled up".



0

$e^{-\beta}(\frac{1}{4})^4$

0

$e^{-3\beta}(\frac{1}{4})^4$

What is the law of S_n under Q_n^β for fixed β and $n \to \infty$?

For $\beta = 0$ we have

$$\begin{array}{rcl} E_P(S_n) & = & nE_P(S_1) & = & 0, \\ \mathrm{Var}_P(S_n) & = & nE_P([S_1]^2) & = & n. \end{array}$$

Hence S_n is of order $\sqrt{n}$. Remarkably, for $\beta \in (0,\infty]$ the behavior depends on the dimension.

Conjecture: *Let* $d \geq 1$ *and* $\beta \in (0,\infty]$. *Then*

$$\mathrm{Var}_{Q_n^\beta}(S_n) \sim Cn^{2\nu} \quad n \to \infty,$$

with front factor $C = C(\beta, d) \in (0,\infty)$ *and exponent* $\nu = \nu(d)$ *given by*

d	1	2	3	4	≥ 5
ν	1	3/4	0.588...	1/2+	1/2

This says that the polymer is:

ballistic	for $d = 1$,
superdiffusive	for $2 \leq d \leq 4$,
diffusive	for $d \geq 5$.

Note that $d = 4$ is a critical dimension.

The conjecture has been proved for:

- $d = 1$ by Greven and dH (1993), with the help of large deviation theory;

- $d \geq 5$ by Brydges and Spencer (1985) and by Hara and Slade (1992), with the help of a perturbative expansion technique called the lace expansion.

It is still open for $2 \leq d \leq 4$, but there is partial progress for:

- $d = 2$ by Lawler, Schramm and Werner ($\geq$ 2003): Stochastic Löwner Evolution with parameter 8/3;

- $d = 4$ by Brydges and Imbrie (2003), on the hierarchical lattice.

The case $d = 3$ remains a mystery.

More precise statements are available:

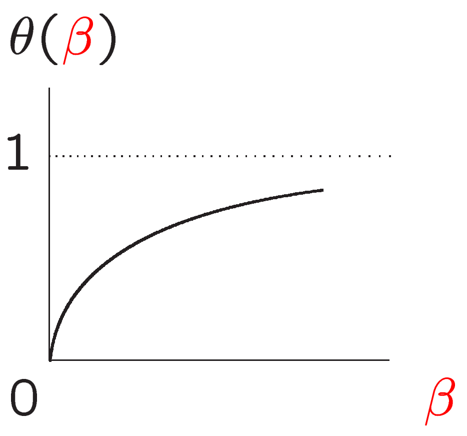
Theorem: (Greven and dH 1993; van der Hofstad and dH 1995; König 1996; van der Hofstad, dH and König 1997)

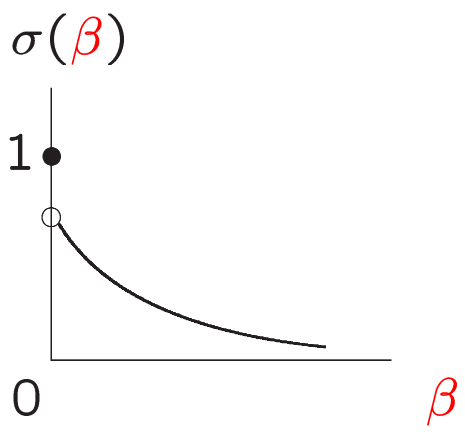
Let $d = 1$. For every $\beta \in (0, \infty)$ there exist numbers $\theta(\beta), \sigma(\beta) \in (0, 1)$ such that as $n \to \infty$

$$Q_n^\beta\left(\frac{|S_n| - \theta(\beta)n}{\sigma(\beta)\sqrt{n}} \in \cdot\,\right) \Rightarrow \mathcal{N}_1(\cdot).$$

For $\theta(\beta), \sigma(\beta)$ variational formulas are known, from which it can be deduced that both are analytic in β and for $\beta \downarrow 0$ behave like

$$\theta(\beta) \sim C_1 \beta^{\frac{1}{3}}, \qquad \sigma(\beta) \sim C_2.$$





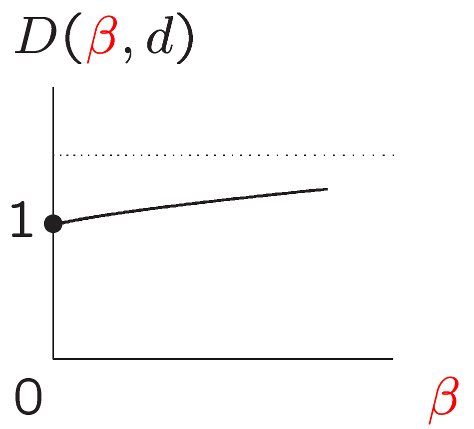
Theorem: (Hara and Slade 1992)

Let $d \geq 5$. *For every* $\beta \in (0, \infty]$ *there exists a number* $D(\beta, d) \in (1, \infty)$ *such that as* $n \to \infty$

$$Q_n^{\beta}\Big(\frac{S_n}{D(\beta, d)\sqrt{n}} \in \cdot\Big) \Rightarrow \mathcal{N}_d(\cdot).$$

For $D(\beta, d)$ *a series expansion is known, from which it can be deduced that*

$$\begin{array}{rclr} D(\beta, d) & = & 1 + O(\beta) & \beta \downarrow 0, \\ D(\infty, d) & = & 1 + O(1/2d) & d \to \infty. \end{array}$$



Note that the last theorem is perturbative in nature while the previous theorem is not.

§ EDWARDS MODEL

The version in continuous space-time of the Domb-Joyce model is the Edwards model, defined informally as

$$\frac{dQ_T^\beta}{dP}(\cdot) = \frac{1}{Z_T^\beta}\, e^{-H_T^\beta(\cdot)},$$

where

$$H_T^\beta((B_t)_{0\le t\le T}) = \beta \int_0^T ds \int_0^T dt\ \delta(B_s - B_t),$$

P denotes the standard Wiener measure on $\mathbb{R}^d$, $T \geq 0$ is the time horizon, $\beta \in [0,\infty)$ is the strength of self-repellence, and $\delta(\cdot)$ is Dirac's function.

A formal definition requires some work.

- $d = 1$: For this case

$$H_T^{\beta}((B_t)_{0\le t\le T}) = \beta \int_{\mathbb{R}} dx\, L(x,T)^2$$

with

$$L(x,t), \qquad t \ge 0,\, x \in \mathbb{R},$$

the Brownian local time process. The integral in the Hamiltonian is called the Brownian self-intersection local time.

Theorem: (van der Hofstad, dH and König 1997)

Let $d = 1$. For every $\beta \in (0,\infty)$, as $T \to \infty$

$$Q_T^{\beta}\Big(\frac{|B_T| - C_1 \beta^{\frac{1}{3}} T}{C_2 \sqrt{T}} \in \cdot\,\Big) \Rightarrow \mathcal{N}_1(\cdot)$$

with C_1, C_2 the same constants that appear in the asymptotics for $\beta \downarrow 0$ of the mean and variance in the CLT for the Domb-Joyce model.

Numerical values are

$$C_1 = 1.11 \pm 0.01, \qquad C_2 = 0.7 \pm 0.1,$$

as follows from a detailed analysis of an associated Sturm-Liouville eigenvalue problem involving the Airy function.

The proof of the CLT for the Edwards model is based on the Ray-Knight representation for Brownian local times in combination with large deviation arguments.

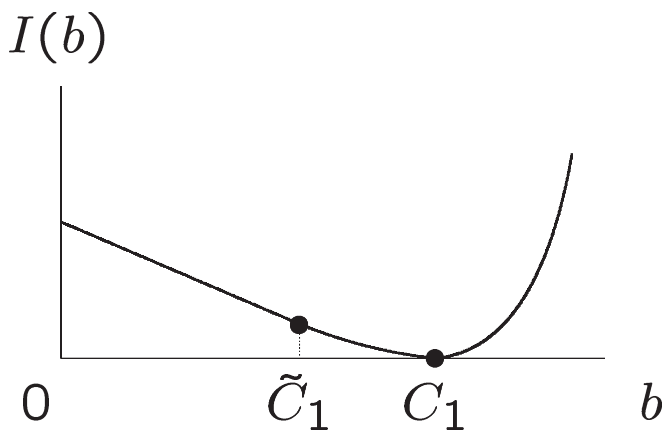
The CLT can be extended to a large deviation principle. This LDP is an "attractor" for a whole class of polymer models in the limit of weak interaction (van der Hofstad, dH and König 2003).

Theorem: (van der Hofstad, dH and König 2003)

Let $d = 1$ and $\beta = 1$. As $T \to \infty$,

$$\frac{1}{T} \log Q_T^{1}\left(\frac{1}{T}|B_T| \approx b\right) = -I(b) \qquad b \in [0, \infty)$$

with a rate function I that looks like:



The linear piece in the rate function is associated with a crossover in the behavior of the path when it is forced to move at a drift that is smaller than some critical drift $\tilde{C}_1 < C_1$.

- $d = 2, 3$:

Brownian local times do not exists when $d \geq 2$. It is possible to define a proper version of Q_T^β, but this requires some renormalization argument. The trouble is that the informal expression for H_T^β has infinite expectation under P.

Varadhan (1969) has carried out the renormalization in $d = 2$ and proved that the resulting path measure is absolutely continuous w.r.t. the Wiener measure. However, not much is known about this path measure, which reflects our lack of understanding of the Domb-Joyce model in $d = 2$.

The situation is worse in $d = 3$. Here the renormalization, which has been carried out by Albeverio, Bolthausen and Zhou (unpublished), is much more difficult. It turns out that the resulting path measure is not absolutely continuous w.r.t. the Wiener measure. Again, we know almost nothing about this path measure.

Conjecture: *In* $d = 2, 3$ *the Edwards model has the same exponent for the mean-square displacement as the Domb-Joyce model.*

- $d \geq 4$:

In this case the Edwards model turns out to be trivial: the renormalized path measure is the Wiener measure. This reflects what we saw for the Domb-Joyce model, which is diffusive in $d \geq 5$ and marginally superdiffusive in $d = 4$.

LECTURE 2:

Variations on basic polymer models

§ ELASTIC POLYMERS

Keep the same definitions as before but change the Hamiltonian to

$$H_n^{\beta,\gamma}((S_i)_{i=0}^n) = \beta \sum_{0\le i<j\le n} 1\{S_i = S_j\}\,(j-i)^{-\gamma}$$

for some $\gamma \in (0,\infty)$. Under the law $Q_n^{\beta,\gamma}$, long loops are less penalized than short loops. The parameter γ is the elasticity.

> Question: How does the elasticity modify the scaling behavior?

Theorem: (Van der Hofstad, dH and Slade 1998)

Let $d \ge 1$. If $\gamma > \max\{\frac{4-d}{2}, 0\}$, then for every $\beta \in (0,\infty)$ the CLT holds under the law $Q_n^{\beta,\gamma}$ with diffusion constant $D(\beta,\gamma,d) \in (1,\infty)$ given by a series expansion.

Note that as γ increases the critical dimension $4-2\gamma$ decreases.

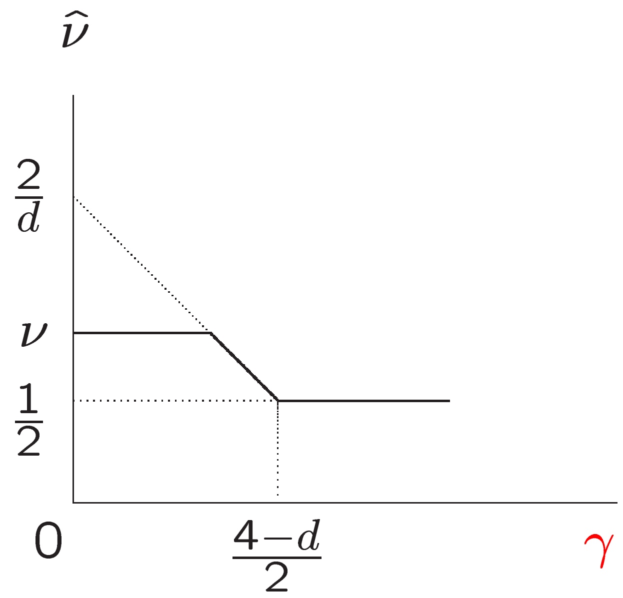
Conjecture: *Let* $1 \le d \le 3$ *and* $\gamma \in (0, \frac{4-d}{2}]$. *For every* $\beta \in (0, \infty)$

$$\mathrm{Var}_{Q_n^{\beta,\gamma}}(S_n) \sim \widehat{C} n^{2\widehat{\nu}} \qquad n \to \infty,$$

with front factor $\widehat{C} = \widehat{C}(\beta, \gamma, d) \in (0, \infty)$ *and exponent* $\widehat{\nu} = \widehat{\nu}(\gamma, d)$ *given by*

$$\widehat{\nu} = \min\{\nu, \tfrac{2-\gamma}{d}\},$$

where $\nu = \nu(d)$ *is the exponent of the soft polymer.*



The conjecture has been proved for $d = 1$, $\gamma \in (0, 1]$ and β sufficiently large (van der Hofstad 2001).

§ CHARGED POLYMERS

This time take Hamiltonian

$$H_n^{\beta,\omega}((S_i)_{i=0}^n) = \beta \sum_{0\le i<j\le n} 1\{S_i = S_j\}\, \omega_i\omega_j,$$

where $\omega = (\omega_i)_{i\in\mathbb{N}_0}$ are i.i.d. random variables with

$$\mathbb{P}(\omega_0 = +1) = p, \quad \mathbb{P}(\omega_0 = -1) = 1 - p.$$

Think of ω_i as the charge of monomer i and of $\mathbb{P}$ as the random charge distribution.

Under the law $Q_n^{\beta,\omega}$, self-intersections of oppositely charged monomers are rewarded while self-intersections of equally charged monomers are penalized.

Question: What is the effect of the random charges on the scaling behavior?

The charge averaged law is

$$\mathbb{Q}_n^{\beta}(\cdot) = \int \mathbb{P}(d\omega)\, Q_n^{\beta,\omega}(\cdot).$$

Theorem: (Biskup and König 2001)

Let $d \geq 1$. If $p = \frac{1}{2}$, then for every $\beta \in (0, \infty)$

$$\mathbb{Q}_n^{\beta}\Big(\frac{S_n}{(n/\log n)^{1/(d+2)}} \in \cdot \Big)$$

converges as $n \to \infty$ to a non-degenerate limit law given by a variational formula.

Thus, when the polymer is electrically neutral it is subdiffusive.

Conjecture: *Let $d \geq 1$. If $p \neq \frac{1}{2}$, then under the law $\mathbb{Q}_n^{\beta}$ the scaling behavior is the same as for the uncharged soft polymer irrespective of β.*

Thus, when the polymer is not electrically neutral it is either superdiffusive ($1 \leq d \leq 4$) or diffusive ($d \geq 5$).

§ DIRECTED POLYMERS IN RANDOM ENVIRONMENT

This time take Hamiltonian

$$H_n^{\beta,\eta}((S_i)_{i=0}^n) = \beta \sum_{i=1}^n \eta(i, S_i),$$

where $\beta \in [0,\infty)$ and

$$\eta = (\eta(i,x))_{i \in \mathbb{N},\, x \in \mathbb{Z}^d}$$

are i.i.d. with a standard Gaussian distribution.

The law $Q_n^{\beta,\eta}$ models a directed polymer

$$(i, S_i)_{i \in \mathbb{N}_0}$$

in $1 + d$ dimensions in a random environment that can be thought of as consisting of "hills" and "valleys".

Theorem: (Carmona and Hu 2002, Comets, Shiga and Yoshida 2003)

(a) If $d = 1, 2$*,* $\beta > 0$ *or* $d \geq 3$*,* $\beta \geq \beta_0(d) > 0$*, then*

$$\limsup_{n\to\infty} \max_{x\in\mathbb{Z}^d} Q_n^{\beta,\eta}(S_n = x) \geq c(d, \beta) > 0 \quad \eta - a.s.$$

(b) If $d \geq 3$*,* $0 \leq \beta \leq \beta_1(d)$*, then*

$$\lim_{n\to\infty} \max_{x\in\mathbb{Z}^d} Q_n^{\beta,\eta}(S_n = x) = 0 \quad \eta - a.s.$$

This result shows that there is a transition from localized to delocalized behavior for the vertical displacement of the polymer as β goes from large to small.

Conjecture: *There is a unique critical value* $\beta_c(d)$ *in* $d \geq 3$*.*

- Delocalized = diffusive
  (Imbrie and Spencer (1988), Bolthausen (1989), Song and Zhou (1996)).

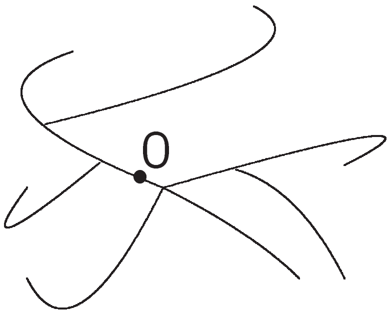
- Localized = superdiffusive
  (Piza (1997), Petermann (2000)).

In the localized regime, the polymer will hunt for the low valleys and concentrate itself there.

In the delocalized regime, the low valleys are not attractive enough for the polymer.

§ BRANCHING POLYMERS

Not all polymers in nature have a chain-like structure. Some of them consist of "many different chains that are knitted together".



One way to model such a polymer is as a lattice tree on $\mathbb{Z}^d$, i.e., a set of sites that is connected, has no loops and contains the origin.

Let $\widehat{Q}_n^\infty$ denote the law of a random lattice tree of size n. Think of this law as the analogue of Q_n^∞, the law of a random SAW of length n.

Question: How far does the polymer extend into space under $\widehat{Q}_n^\infty$ as $n \to \infty$?

Theorem: (Hara and Slade 1990; Derbez and Slade 1997)

Let $d \geq 9$. Under $\widehat{Q}_n^\infty$, if each monomer is given a mass n^{-1} and space is shrunk by a factor $n^{\frac{1}{4}}$, then the branching polymer converges to integrated super-Brownian excursion (ISE) conditioned to have total mass 1. The diffusion constant $D(d) \in (1, \infty)$ of the underlying Brownian motion is given by a series expansion.

ISE is an example of a dissipative superprocess: a mass distribution on $\mathbb{R}^d$ that evolves in time and eventually dies out. It shows up as the scaling limit in several other spatially interacting models (Hara, van der Hofstad and Slade 2003) and plays a role analogous to the normal distribution in the CLT.

Conjecture: *Let $1 \leq d \leq 8$. Under $\widehat{Q}_n^\infty$ the branching polymer is more extended than order $n^{\frac{1}{4}}$.*

LECTURE 3:

Heteropolymers

§ HETEROPOLYMERS NEAR A SINGLE INTERFACE

This time change H_n^β to

$$H_n^{\beta,h,\omega}((S_i)_{i=0}^n) = \beta \sum_{i=1}^n (\omega_i + h)\Delta_i,$$

where $\omega = (\omega_i)_{i\in\mathbb{N}}$ are i.i.d. with

$$\mathbb{P}(\omega_1 = +1) = \mathbb{P}(\omega_1 = -1) = \frac{1}{2},$$

$h \in [0,1]$ is a bias parameter, and

$$\begin{aligned} \Delta_i &= +1 \quad \text{if } (S_{i-1}, S_i) \geq 0, \\ \Delta_i &= -1 \quad \text{if } (S_{i-1}, S_i) \leq 0. \end{aligned}$$

The law $Q_n^{\beta,h,\omega}$ models the following situation. The

$$\begin{aligned} \text{upper half plane} &= \text{oil} \\ \text{lower half plane} &= \text{water}. \end{aligned}$$

The directed path $(i, S_i)_{i\in\mathbb{N}_0}$ describes a polymer chain whose monomers point right and up/down. Monomer i is

$$\begin{array}{ll} \text{hydrophobic} & \text{if } \omega_i = +1 \\ \text{hydrophilic} & \text{if } \omega_i = -1. \end{array}$$

Monomer i lies in the

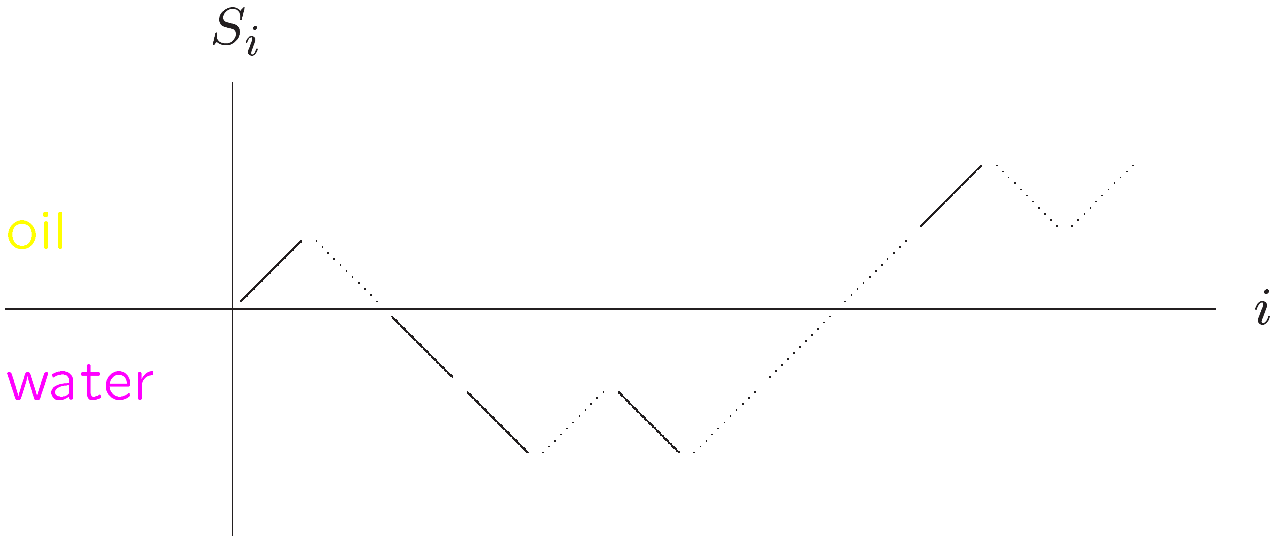
$$\begin{array}{ll} \text{oil} & \text{if } \Delta_i = +1 \\ \text{water} & \text{if } \Delta_i = -1. \end{array}$$

Thus, the interaction rewards matches and penalizes mismatches in the chemical affinities. The parameter h is an affinity bias.

$n = 12$

$\omega_i = +1$ for $i = 2, 5, 7, 8, 9, 11, 12$

$\omega_i = -1$ for $i = 1, 3, 4, 6, 10$



$$H_n^{\beta,h,\omega} = -\beta(4-3)(+1+h) - \beta(3-2)(-1+h)$$

The interesting aspect of this model is that the polymer feels a competition between

– staying close to the interface, so as to place more monomers in their preferred medium (localised)

– wandering away from the interface, so as to explore more space (delocalised).

In essence, this is a competition between energy and entropy.

$\boxed{\text{Which of the two strategies wins?}}$

Subadditivity arguments show that, for every β and h, the free energy

$$\phi(\beta, h) = \lim_{n\to\infty} \frac{1}{n} \log Z_n^{\beta,h,\omega}$$

exists and is constant ω-a.s. Moreover,

$$\phi(\beta, h) \geq \beta h$$

with equality when the polymer wanders away to $-\infty$. Indeed, if $\Delta_i = +1$ for $i \to \infty$, then $H_n^{\beta,h,\omega} = -\beta h n + o(n)$ for $n \to \infty$.

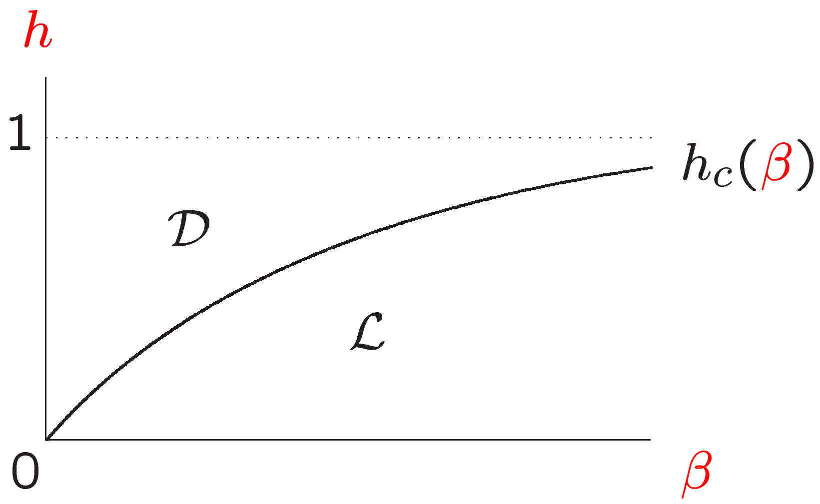
Define

$$\begin{array}{rcl} \mathcal{L} & = & \{(\beta, h) \colon \ \phi(\beta, h) > \beta h\} \\ \mathcal{D} & = & \{(\beta, h) \colon \ \phi(\beta, h) = \beta h\}. \end{array}$$

The polymer is said to be localised in $\mathcal{L}$ and delocalised in $\mathcal{D}$.

Theorem: (Bolthausen and dH 1997)

There exists a phase transition curve $\beta \mapsto h_c(\beta)$ *separating the two phases:*



Moreover, this curve is continuous and non-decreasing, and satisfies

$$\begin{array}{ll} h_c(\beta) \sim C_1\beta & \beta \downarrow 0 \\ 1 - h_c(\beta) \sim C_2/\beta & \beta \to \infty. \end{array}$$

The proof of this theorem is based on large deviation estimates in combination with perturbation arguments. It is hard to get control over $h_c(\beta)$.

More precise information is available.

Theorem: (Biskup and dH 1999)

(a) For $(\beta, h) \in \mathcal{L}$*, the polymer crosses the interface with a strictly positive frequency, and its excursions are exponentially tight.*

(b) For $(\beta, h) \in \mathcal{D}$*, the polymer crosses the interface with a zero frequency.*

The proof is based on Gibbs theory in combination with coupling arguments.

Conjecture: *For* $(\beta, h) \in \mathrm{int}(\mathcal{D})$*, the polymer crosses the interface finitely often and afterwards escapes to* $+\infty$*.*

There are important recent developments concerning the slope of the phase transition curve at $\beta = 0$.

Conjecture: (Monthus 2000)

$$h_c(\beta) = \frac{1}{4\beta/3} \log\cosh(4\beta/3).$$

The argument of Monthus is based on a general renormalization scheme, which however is non-rigorous.

Theorem: (Bodineau and Giacomin 2004)

$$h_c(\beta) \geq \frac{1}{4\beta/3} \log\cosh(4\beta/3).$$

This lower bound is the complement of the upper bound

$$h_c(\beta) \leq \frac{1}{2\beta} \log\cosh(2\beta)$$

that was derived in Bolthausen and dH (1997) with the help of a simple annealed estimate. Consequently,

$$\frac{2}{3} \leq \lim_{\beta \downarrow 0} \frac{1}{\beta} h_c(\beta) \leq 1.$$

It is believed that the slope equals $\frac{2}{3}$ and is insensitive to the details of the model: universality.

In Bodineau and Giacomin (2004), ω is i.i.d. with an arbitrary marginal distribution that is symmetric and has an exponentially bounded tail.

§ HETEROPOLYMERS NEAR MULTIPLE INTERFACES

Instead of one interface, it is also interesting to consider a multi-interface model, where the water and the oil occur in alternating horizontal layers:

oil

water

oil

water

The same $H_n^{\beta,h,\omega}$ as above applies, but now with $\Delta_i = \pm 1$ according as monomer i lies in an oil-layer or in a water-layer.

The polymer will hop between neighboring interfaces at a rate that depends on β, h and on the layer width.

What is the hopping rate?

Suppose that all the layers have equal width d_n, which depends on the length n of the polymer.

Theorem: (dH and Wüthrich 2004)

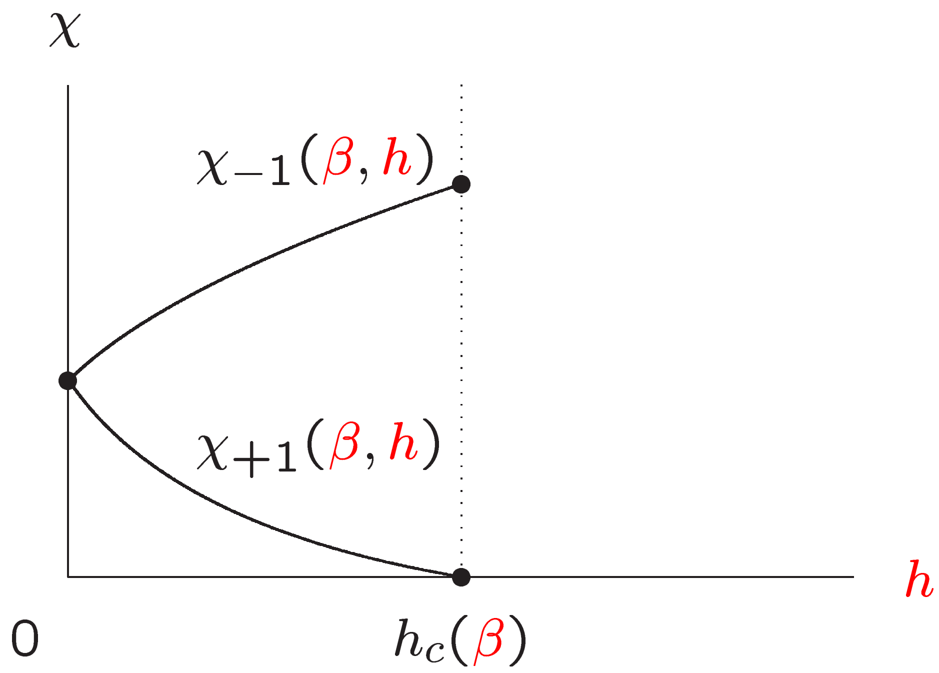
(a) If $\lim_{n\to\infty} d_n = \infty$*, then the free energy is the same as for the single interface.*

(b) Let $(\beta, h) \in \mathcal{L}$*. Suppose that*

$$\begin{aligned} \lim_{n\to\infty} d_n/\log\log n &= \infty \\ \lim_{n\to\infty} d_n/\log n &= 0. \end{aligned}$$

Then there exist curves $h \mapsto \chi_{\pm 1}(\beta, h)$ *such that the* ω*-averaged time for the polymer to cross a layer is*

$$\begin{array}{ll} e^{d_n \chi_{+1}(\beta,h)[1+o(1)]} & \textit{for oil} \\ e^{d_n \chi_{-1}(\beta,h)[1+o(1)]} & \textit{for water.} \end{array}$$



The proof is based on decoupling and reflection arguments. There is no good control over $\chi_{+1}(\beta, h), \chi_{-1}(\beta, h)$.

Conjecture: $h \mapsto \chi_{-1}(\beta, h)$ has a smooth continuation across $h_c(\beta)$.

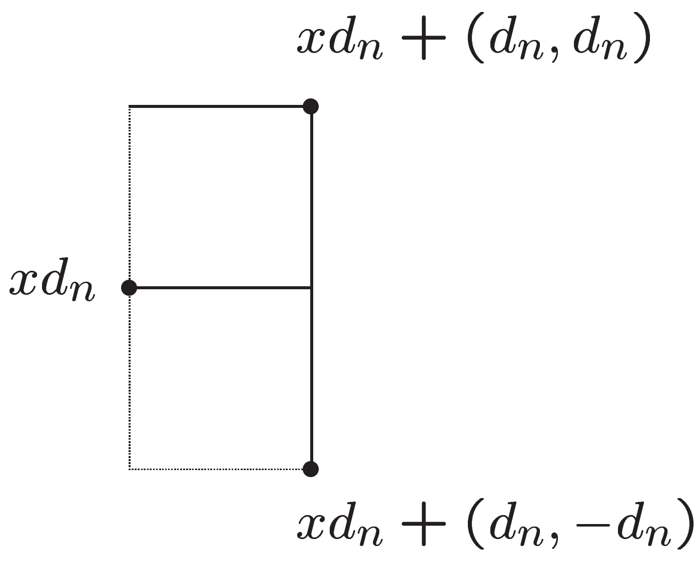
§ HETEROPOLYMERS IN EMULSIONS

What can be done for more complex interface geometries?

Divide $\mathbb{Z}^2$ into square blocks of size d_n. Each block is

oil	with probability p
water	with probability $1-p$.

The polymer follows a directed path where steps up, down and to the right are allowed. In order to make the model tractable, we allow paths to enter and exit the blocks only at opposite corners:



The interaction Hamiltonian is

$$H_{n,d_n}^{\omega,\Omega}((S_i)_{i=0}^n) = -\alpha M_{+1} - \beta M_{-1},$$

with $0 \leq \beta < \alpha < \infty$ interaction strengths,

$$\begin{aligned} M_{+1} &= \sum_{i=1}^{n} 1\{\omega_i = \Omega_{d_n}(i, S_i) = +1\} \\ M_{-1} &= \sum_{i=1}^{n} 1\{\omega_i = \Omega_{d_n}(i, S_i) = -1\}, \end{aligned}$$

where, as before,

$\omega_i = +1$ if monomer i is hydrophobic
$\omega_i = -1$ if monomer i is hydrophilic,

but now

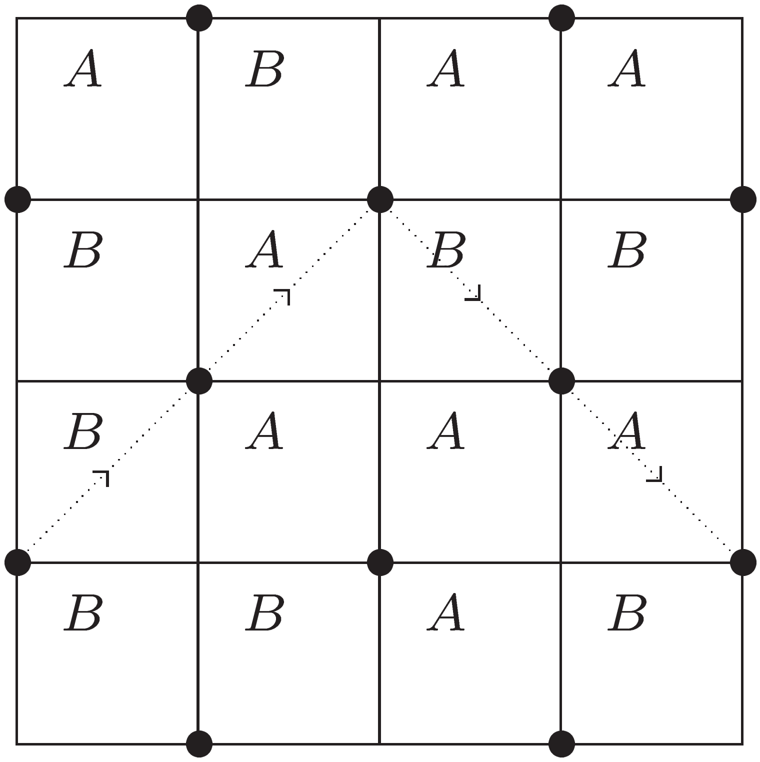
$\Omega_{d_n}(i, S_i) = +1$ if (i, S_i) lies in oil
$\Omega_{d_n}(i, S_i) = -1$ if (i, S_i) lies in water.

Here,

$$\Omega = (\Omega_{d_n}(x))_{x \in \mathbb{Z}^2}$$

is a second random ingredient, describing the percolation structure of the blocks.

Note that matches are rewarded while mis-matches or not penalised.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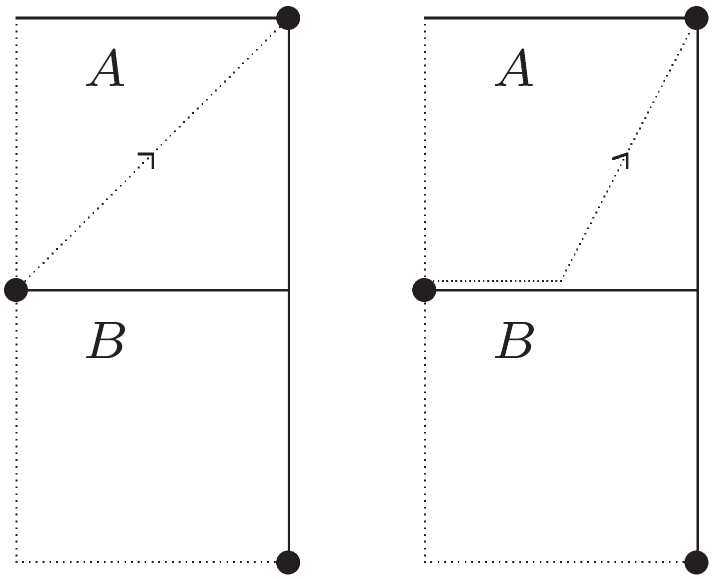
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Theorem: (dH and Whittington, in progress)

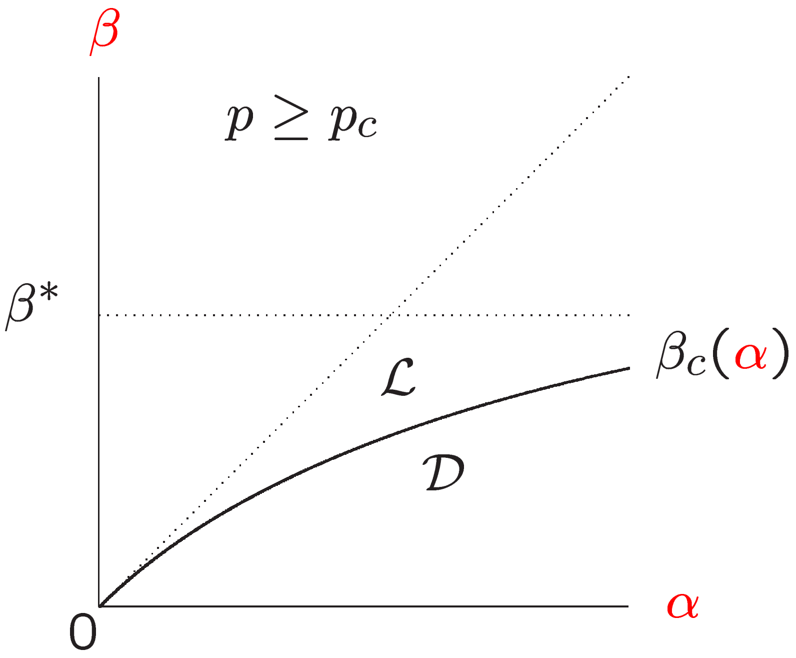
(a) Assume that

$$\begin{aligned} \lim_{n\to\infty} d_n &= \infty \\ \lim_{n\to\infty} d_n/n &= 0. \end{aligned}$$

Then the free energy exists, is constant ω, Ω*-a.s. and does not depend on the choice of* (d_n)*.*

(b) For every $p \in (0,1)$*, there exists a phase transition curve* $\alpha \mapsto \beta_c(\alpha; p)$ *separating* $\mathcal{L}$ *from* $\mathcal{D}$*.*

(c) Let $p_c \approx 0.64$ *be the critical threshold for directed site percolation on* $\mathbb{Z}^2$*. Then for* $p \geq p_c$ *the phase transition curve is independent of* p*, while for* $p < p_c$ *it depends on* p *and moves down with* p*.*



(d) For $p \geq p_c$ the phase transition curve is continuous, non-decreasing and concave, and satisfies

$$
\begin{array}{ll}
\alpha - \beta_c(\alpha) \asymp \alpha^2 & \alpha \downarrow 0, \\
\beta_c(\alpha) \to \beta^* \in [\log 2, 8 \log 3) & \alpha \to \infty.
\end{array}
$$

Interpretation:

For $p \geq p_c$, the oil (almost) percolates:

- In $\mathcal{L}$, the polymer moves along the oil-water interface part of the time.

- In $\mathcal{D}$, the polymer moves inside the oil all the time.

For $p < p_c$, the oil does not percolate. Hence the polymer must hop between oil droplets. Therefore the phase transition curve is more complicated.