One dimensional models with random magnetic fields

The contribution of Enza Orlandi

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Works in collaboration with

- Marzio Cassandro (Gran Sasso Science Institute l'Aquila) ;
- Enza Orlandi (Roma TRE);
- Maria Eulalia Varès (Rio de Janeiro).

With Enza we published 6 papers together.
We start working together with Enza & Marzio around 1996-97.
At that moment, I was finishing to work with Anton Bovier and Véronique Gayrard on an extension to the Hopfield model (a mean field model of spin-glasses) of a very important paper by

Cassandro, Orlandi & Presutti
Interfaces and typical Gibbs configurations for one dimensional Kac potentials
Prob. Theory Relat. Fields Vol 96, 57-96, 1993.

which was an extension to infinite volume of a part of the paper by

Francis Comets
Nucleation for a long range magnetic model. Ann. Inst. H. Poincaré, vol 23,2,135-178 (1987)

It happens that the case of the Hopfield-Kac model was too complicated and we decided to consider as a warming exercice the simplest case of two patterns which is in fact the random field model.

Leaving for the future the Hopfield case.

We thought at that moment (1997-1998) that this will be easy.

In fact we were involved in this problem until 2009.

We publish 4 papers on the one dimensional Random Field Kac model with a total of 195 pages.

- The first was published in 1999 :

M.Cassandro, E.Orlandi, P. Picco: " Typical configurations for one dimensional random field Kac model " Ann. Prob., 27, No. 3, 1414-1467, 1999

It is an almost sure result (almost surely with respect to the realisations of the random magnetic fields) and we prove basically that by a kind of Imry & Ma argument that the fluctuations of the random fields on a scale $1/\gamma^2$, destroy the ferromagnetic order that occurs without magnetic fields on a scale $e^{+\mathcal{F}/\gamma}$. Here $1/\gamma$ is the length of interactions in the Kac model. These fluctuations correspond to the ones that come from the Law of the Iterated Logarithm when sampling along disjoint intervals of convenient length that are $1/\gamma^2$.

We had at that moment (1999) a lot of discussions with Errico Presutti that did not like to much our result because we prove that there are fluctuations but we were unable to answer to the simplest question :

Tell me what is the volume around the origin such that if I give you the realisation of the random fields there, you tell me what is the empirical magnetisation profiles around the origin ?

Some weeks after, Errico gave us two pages of computations, of what should be a random functional (of large deviation) that will govern the system. It was based on properties of the Brownian motion published in an article he did not remember who wrote it.

It was clear at that moment that we needed a more precise result on the way the interface profiles happen.
So we extend a part of the results of

A. De Masi, E. Orlandi, E. Presutti, L. Triolo:" Uniqueness of the instanton profile and global stability in non local evolution equation, " Rendiconti di Matematica. Serie VII, vol 14, 693-723, 1994

that consider a single differential equation (non-linear and non-local for a function from $\mathbb{R}$ to $\mathbb{R}$) to a situation of system of two coupled equations for a vector valued function from $\mathbb{R}$). Since comparison principles where heavily used in that paper a priori the extension to such a coupled system seems quasi-impossible.
This was done in

M.Cassandro, E.Orlandi, P. Picco : " The optimal interface profile for a nonlocal model of phase separation ", Nonlinearity 15, pag 1621-1651 (2002)

To answer the Errico's questions, and also another remark from H.T Yau about the possible values of the parameter θ in front of the magnetic fields a lot of work was needed.
First extend the result of a representation to all the values of θ and β were the "canonical" free energy of Random Field Curie-Weiss Model (RFCW) has two absolute minima.
For what concerns the phase diagram of the RFCW model see also the paper by Christoph & Arnaud pg 440

Cristoph Külske & Arnaud Le Ny ;
Spin-Flip Dynamics of the Curie-Weiss Model: Loss of Gibbsianness with Possibly Broken Symmetry Commun. Math. Phys. 271, 431–454 (2007)

Then abandon the idea to have an almost sure result but just a result that will have a probability that goes to 1 when $\gamma \downarrow 0$

We did this in the paper

M.Cassandro, E.Orlandi, P. Picco, M.-E. Vares : " One-dimensional random field Kac's model:localization of the phases. ", Electronic Journal Probability, 20, pages 786-864 (2005)

Let us mention that we were able to do it using in particular two ingredients

- a concentration inequality for Lipschitz function of Bernoulli random variables (the random magnetic fields)
- a cluster expansion to estimate the corresponding Lipschitz norms.

These two arguments are classical in two different communities.

At that moment we gave an answer to the Errico's question above and proved that the volume needed to know the empirical magnetisation around the origin (in a block of size $1/\gamma$) we need a volume like $[-Q(\gamma)/\gamma^2, +Q(\gamma)/\gamma^2]$ where Q is given by the nice function

$$(\log(1/\gamma))^{\frac{1}{\log\log\log(1/\gamma)}}$$

since the paper is 78 pages long, I will not comment too much on it.

Around that period (2005), we discussed with various persons and discover with the help of Jean-François Le Gall that the mysterious paper of Errico was in fact a paper of Neveu and Pitman at the Séminaire de Probabilité (Strasbourg) published in 1989. We also had illuminating discussions with Jean Bertoin and Isaco Meilijson on the Neveu-Pitman paper.

Meanwhile Marzio & Eulalia gave up. We wrote:

E.Orlandi and P. Picco " One-dimensional random field Kac's model: weak large deviations principle " Electronic Journal Probability, Vol 14, pages 1372-1416 (2009).

We exhibit the typical configuration of the RFKM and we prove a weak large deviation principle (in that order as it should be).
In fact the typical configurations can be described as follows : The limiting distribution (when $\gamma \downarrow 0$) of the inter-distance between jump points (between two changes of phases) (this with respect to the distribution of the random magnetic fields) is the Neveu-Pitman stationnary renewal process of h-extrema of a bilateral Brownian motion.

This is also linked to drawback in finance (Isaco), also to valleys in the Sinai random walk in random environment (Sinai, Kesten, Kolosov).

Other applications of Neveu-Pitman stationary renewal process of h-extrema of a bilateral Brownian motion can be found in :

Bovier & Faggionato
Spectral analysis of Sinai's walk for small eigenvalues
Ann. Probab. Volume 36, Number 1 (2008), 198-254.

We also proved that the random functional Errico Presutti wrote for us 5 years before was the true one.

At that moment (around 2004-2005) Errico and Marzio were involved with Pablo Ferrari and Titti Merola in a re-reading of the Fröhlich & Spencer article on one-dimensional Ising model with long range interaction, they published :

M. Cassandro, P. A. Ferrari, I. Merola and E. Presutti:
Geometry of contours and Peierls estimates in $d = 1$ Ising models with long range interaction.
J. Math. Phys. 46, no 5, 053305, (2005)

So it was natural to consider the case of the one-dimensional random field Ising model with long range interaction.

A priori, it seems a lot more complicated that the not so simple Bricmont & Kuipianen 1988 article due to the long range. As Jurg Fröhlich said to Errico.
In fact after a lot of computations, simplifications, and unexpected cancellations etc., it happens to be quasi simple and in

M. Cassandro, E.Orlandi and P. Picco: " Phase Transition in the 1d Random Field Ising Model with long range interaction. " Comm. Math. Phys, vol Issue 2,pages 731-744 (2009) we prove that for the decay of the interaction is

$$\frac{1}{r^{2-\alpha}}$$

with $\alpha \in]1/2, (\log 3/\log 2) - 1]$, if the temperature is small enough, and the parameter θ in front of the magnetic field is small enough then then there is at least two Gibbs states.

We are sure that this happen for $\alpha \in]1/2, 1[$ and we are working on it with my chilian student Jorge Littin.

Unfortunately we cannot use our proof to simplify the Jean Bricmont & Antti Kuipianien article for the three dimensional Random Field Ising model with nearest neighbour interaction (1988).

Note that in 2013, Leuzzi & Parisi studied the same model (at the level of ground states) and predicted using computer simulations our results published in 2009, which means that they are very smart as everybody knows.

L. Leuzzi and G. Parisi
Imry-Ma criterion for long-range random field Ising model: short-/long-range equivalence in a field
(March 27, 2013) (Arxiv)

and what should be the published version :

L. Leuzzi and G. Parisi
Long-range random-field Ising model: Phase transition threshold and equivalence of short and long ranges Phys. Rev. B 88, 224204 (2013) - Published 19 December 2013

Then with Enza and Marzio, we wanted to understand what's happen in the case $0 \leq \alpha \leq 1/2$ where no phase transition occurs as proved Aizenmann and Wehr in 1989. Our last paper with Enza is

M. Cassandro, E.Orlandi and P. Picco: " Typical Gibbs configurations for the 1d Random Field Ising Model with long range interaction. " Comm. Math. Phys.: Vol 309, Issue 1, Page 229-253, (2012) DOI: 10.1007/s00220-011-1371-1

We proved that typical configurations are made of alternating runs of $+1$ and -1

when $\alpha = 0$ the length of the runs $\mathcal{L}_i$ satisfies

$$c_1(0) \leq \theta^2 |\mathcal{L}_i| \leq c_2(0)(\log(1/\theta))^3$$

over volumes of diameter

$$\frac{1}{\theta^2} \times \frac{1}{\theta} \log(1/\theta)$$

with $\mathbb{P}$-probability larger than $1 - \frac{\theta}{\log(1/\theta)}$
and Gibbs measure larger than $1 - \frac{\theta}{\log(1/\theta)}$

When $0 < \alpha < 1/2$ the length of the runs $\mathcal{L}_i$ satisfies

$$c_1(\alpha) \left(\log \frac{1}{\theta}\right)^{-\frac{2}{1-2\alpha}} \left(\log \log \frac{1}{\theta}\right)^{-\frac{1}{1-2\alpha}} \leq \theta^{\frac{2}{1-2\alpha}} |\mathcal{L}_i|$$

and

$$\theta^{\frac{2}{1-2\alpha}} |\mathcal{L}_i| \leq c_2(\alpha) \left(\log \frac{1}{\theta}\right) \left(\log \log \frac{1}{\theta}\right)$$

over volumes of diameters

$$c_0(\alpha) e^{\log(1/\theta) \log \log(1/\theta)} \left(\frac{1}{\theta}\right)^{\frac{2}{1-2\alpha}}$$

with $\mathbb{P}$-probability larger than $1 - e^{-\log(1/\theta) \log \log(1/\theta)}$
and Gibbs measure larger than $1 - e^{-\log(1/\theta) \log \log(1/\theta)}$

When $\alpha = 1/2$ our results are weaker, we prove that the runs that contains the origin satisfies

$$\frac{c_1}{\theta} \leq \log |\mathcal{L}_1| \leq \frac{c_2}{\theta^2}$$

with $\mathbb{P}$-probability larger than $1 - e^{-\frac{c_0}{\theta^2}}$
and Gibbs measure larger than $1 - e^{-\frac{c_0}{\theta^2}}$

In fact we have better estimates for the diameter of the volumes where the upper bound is valid. To have a lower and a upper bound together we have the constraint that comes from the lower bound.

We had the project to give more accurate estimates, like a convergence in distribution to a kind of renewal process associated to the fractional Brownian motion and then prove a large deviations principle as we did for the Random Field Kac model.

This was supposed to be done during my last stay in Rome (September 2015-August 2016).

We didn't do it.

THE END

This is the end, beautiful friend
This is the end, my only friend, the end
Of our elaborate plans, the end
Of everything that stands, the end
No safety or surprise, the end
I'll never look into your eyes, again.

(The Doors, 1967)