Stochastic particle systems, hydrodynamic limits and free boundary problems.

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Hydrodynamic limit for stochastic particle systems confined in a region.

PDE's have to be complemented with the boundary conditions.

• Boundary effects are determined by the forces acting to keep the system confined in a bounded region.
Most studied case: boundary forces are due to reservoirs which fix the densities at the boundaries.

• Free boundary problems: **region confining the system is determined by the state of the system itself.**

Macroscopic theory and examples of microscopic models.

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Macroscopic theory and examples of microscopic models.

Macroscopic theory

The systems are continuum bodies confined in a region Ω, each point $r \in \Omega$ is representative of a large microscopic system.

I will consider only one dimension and systems with a unique *order parameter* (density).

Macroscopic states are non negative $L^1(\Omega)$ **functions** $\rho(r)$ $r \in \Omega$ (the summability assumption ensures that the total mass $\displaystyle\int_\Omega \rho(r)$ is well defined.)

Postulate: the thermodynamics of the system is determined by a free energy functional $F(\rho)$.

Equilibrium thermodynamical states are the minima of the free energy functional

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Macroscopic theory: evolution

The first equation is the law of conservation of mass:

$$\frac{\partial \rho}{\partial t} = -\frac{\partial J}{\partial r}$$

with $J = J(r, t)$ the current.
The above continuity equation has to be complemented with a constitutive equation for the current. The choice is finalized to ensure decrease of the free energy:

$$J = -\kappa(\rho)\frac{\partial}{\partial r}\left(\frac{\delta F(\rho)}{\delta \rho(r)}\right)$$

$\kappa(\rho) > 0$ is a model dependent coefficient called *mobility*.

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Macroscopic theory: periodic boundary conditions

Remark: with periodic b.c. we avoid interaction with walls!

$$\frac{\partial \rho}{\partial t} = -\frac{\partial J}{\partial r}, \qquad J = -\kappa(\rho)\frac{\partial}{\partial r}\left(\frac{\delta F(\rho)}{\delta \rho(r)}\right), \qquad r \in \Omega$$

Assume Ω is the unit circle, then the total mass is conserved

$$\frac{d}{dt}\int_\Omega \rho(r,t)dr = 0$$

and the free energy is monotone non increasing

$$\frac{dF(\rho(\cdot,t))}{dt} = \int_\Omega \frac{\delta F(\rho)}{\delta \rho(r)}\;\frac{\partial \rho}{\partial t}\;dr = -\int_\Omega \kappa\left(\frac{\partial}{\partial r}\frac{\delta F(\rho)}{\delta \rho(r)}\right)^2 dr \leq 0$$

(integrating by parts and using periodicity)

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Macroscopic theory: periodic boundary conditions. Example.

Free energy is the entropy: $F(\rho) = \int_\Omega f(\rho(r))dr$

$$f(\rho) = \rho \log \rho + (1-\rho)\log(1-\rho)$$

Its gradient flow is

$$\frac{d\rho}{dt} = \frac{d}{dr}\Big(\kappa(\rho)\frac{d}{dr}\log\frac{\rho}{1-\rho}\Big)$$

which, with the choice $\kappa(\rho) = \frac{1}{2}\rho(1-\rho)$ becomes the heat equation

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Microscopic model for the example (periodic boundary conditions)

Symmetric exclusion process on $\Lambda_\varepsilon := \varepsilon^{-1}\Omega \cap \mathbb{Z}$, Ω the circle.

$\{\eta_t(x) \in \{0,1\}, x \in \Lambda_\varepsilon, t \geq 0\}$ is the process with generator:

$$L_0 f(\eta) = \frac{1}{2} \sum_{x \in \Lambda_\varepsilon} \sum_{y:|y-x|=1} \Big(f(\eta^{(x,y)}) - f(\eta) \Big)$$

Invariant measures are ν_ρ product of Bernoulli, formally given by the Gibbs formula

$$\nu_\rho(\eta) = \prod_x \exp\left\{ \frac{1}{2} [\eta(x) \log \rho + (1-\eta(x)) \log(1-\rho)] \right\}$$

The mobility is

$$\kappa(\rho) = \frac{1}{2} \sum_x \nu_\rho \Big(\eta(0)[\eta(x) - \rho] \Big) = \frac{1}{2} \rho(1-\rho)$$

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Symmetric exclusion process: hydrodynamic limit (periodic boundary conditions)

$\Lambda_\varepsilon := \varepsilon^{-1}\Omega \cap \mathbb{Z}$, Ω the circle.

Order parameter (empirical averages): $\ell = \varepsilon^{-b}$, $b \in (0,1)$

$$\mathcal{M}_\ell(r,\eta) := \frac{1}{\ell} \sum_{x:|x-\varepsilon^{-1}r|\leq \ell} \eta(x), \qquad r \in \Omega$$

Initial conditions: $\rho_0(r) \geq 0$ $r \in \Omega$ fixed. **The law of** η_0 **approximates the initial profile** ρ_0

$$\lim_{\varepsilon\to 0} P\Big(\sup_r |\mathcal{M}_\ell(r,\eta_0) - \rho_0(r)| \leq \varepsilon^a\Big) = 1$$

$a > 0$ small.

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Symmetric exclusion process: hydrodynamic limit (periodic boundary conditions)

Theorem

Given any $T > 0$, for all $t \leq T$

$$\lim_{\varepsilon \to 0} P\Big(\sup_r |\mathcal{M}_\ell(r, \eta_{\varepsilon^{-2}t}) - \rho(r,t)| \leq \varepsilon^a \Big) = 1$$

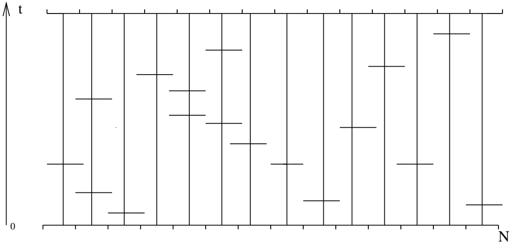
with $\rho(r,t)$ solution of the heat equation:

$$\frac{\partial \rho}{\partial t} = \frac{1}{2} \frac{\partial^2 \rho}{\partial r}, \qquad r \in \Omega$$

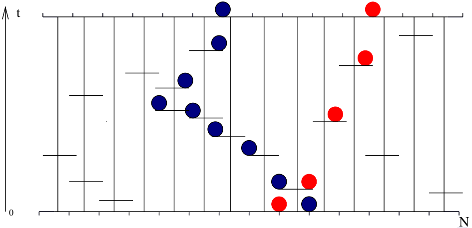
with initial condition $\rho(r,0) = \rho_0$.

Stirring.

At each pair of n.n. sites **Poisson clock of intensity** $\frac{1}{2}$,



when it rings **exchange the occupation numbers.**



Symmetric exclusion process: Fick's law (periodic boundary conditions)

The **macroscopic current** $J(r,t)$ satisfies the **Fick's law**:

$$J(r,t) = -\frac{1}{2}\frac{\partial\rho(r,t)}{\partial r}$$

Microscopic current is the expected signed mass crossing a point $x+\frac{1}{2}$ per unit time (from the left minus that from the right).

$$j(x,\eta_t) := \frac{1}{2}\left[\eta_t(x) - \eta_t(x+1)\right]$$

$[\cdot]$ the integer part, $r \in \Omega$ and $t \leq T$

$$\lim_{\varepsilon\to 0}\varepsilon^{-1}\ \mathbb{E}\Big(j([\varepsilon^{-1}r],\eta_{\varepsilon^{-2}t})\Big) = -\frac{1}{2}\frac{\partial\rho(r,t)}{\partial r}$$

At equilibrium current=0.

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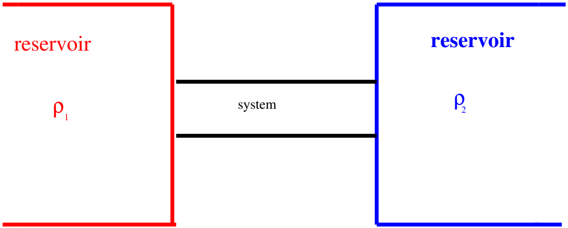
Open systems

Open means that the system is in contact with the "outside".

"Typical example": a metal bar cooled at one end and warmed at the other, the two extremes being kept at two different temperatures $T_+ > T_-$.

In our set up we consider densities, so the system is in contact with two reservoirs that keep the densities equal to ρ_1 in one side and to ρ_2 in the other side.



Macroscopic theory: Dirichlet boundary conditions

$\Omega = [0,1]$ and $F(\rho) = \int_\Omega f(\rho(r))dr$ is the free energy.

Natural to complement the equation with Dirichlet b. c.:

$$\frac{\partial \rho}{\partial t} = -\frac{\partial J}{\partial r}, \qquad J = -\kappa(\rho)\frac{\partial f'(\rho)}{\partial r}, \quad r \in (0,1)$$

$$\rho(0,t) = \rho_0, \qquad \rho(1,t) = \rho_1, \quad \rho(r,0) \text{ given}$$

The **total mass is not conserved:**

$$\frac{d}{dt}\int_0^1 \rho(r,t)dr = J(0,t) - J(1,t)$$

The **free energy is not monotone:**

$$\frac{dF(\rho(\cdot,t))}{dt} = J(0,t)f'(\rho_0) - J(1,t)f'(\rho_1) - \int_0^1 \kappa(\rho)\Big(\frac{\partial f'(\rho(r,t))}{\partial r}\Big)^2 dr$$

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Law of thermodynamics: *the free energy is monotone non increasing*.

$$\int_0^1 \rho(r,t)dr = \int_0^1 \rho_0(r)dr + X_0(t) - X_1(t)$$

$$X_0(t) = \int_0^t J(0,s)ds, \qquad X_1(t) = \int_0^t J(1,s)ds$$

Interpretation: the reservoir connected at 0 send in a mass $X_0(t)$, the reservoir connected at 1 remove a mass $X_1(t)$.

Λ_0= region occupied by the left reservoir, Λ_1= region occupied by the right reservoir

Assume: $|\Lambda_0|$ and $|\Lambda_1|$ very large and that the reservoirs "instantaneously" homogeinize any change of mass

Macroscopic theory: Dirichlet boundary conditions

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Macroscopic theory: Dirichlet boundary conditions

Left reservoir: at t=0 has density ρ_0, at time t has density

$$\rho_0 - \frac{X_0(t)}{|\Lambda_0|} \approx \rho_0$$

Right reservoir: at time t has density $\rho_1 + \dfrac{X_1(t)}{|\Lambda_1|} \approx \rho_1$

The free energies at time t are

$$F_{\Lambda_0,t} = |\Lambda_0| f(\rho_0 - \frac{X_0(t)}{|\Lambda_0|}) \approx F_{\Lambda_0,0} - f'(\rho_0) X_0(t)$$

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The total free energy

$$
\begin{aligned}
F^{\text{total}} &= F(\rho(\cdot,t)) + F_{\Lambda_0,t} + F_{\Lambda_1,t} \\
&\approx F(\rho(\cdot,t)) + F_{\Lambda_0,0} - f'(\rho_0)J(0,t) + F_{\Lambda_1,0} + f'(\rho_1)J(1,t)
\end{aligned}
$$

is monotone non increasing:

$$
\begin{aligned}
\frac{dF^{\text{total}}}{dt} &= \frac{dF(\rho(\cdot,t))}{dt} - f'(\rho_0)X_0(t) + f'(\rho_1)X_1(t) \\
&= -\int_0^1 \kappa(\rho)\Big(\frac{\partial f'(\rho(r,t))}{\partial r}\Big)^2\, dr
\end{aligned}
$$

Symmetric exclusion process: density reservoirs

SSEP in $\Lambda_\varepsilon = [0, \varepsilon^{-1}] \cap \mathbb{Z} = \{0, 1, ..., N\}, \quad \mathbf{N} = [\varepsilon^{-1}]$.

Put two independent **Poisson clocks of intensity** $\frac{1}{2}$ at the pairs $(-1, 0)$ and $(N, N+1)$.

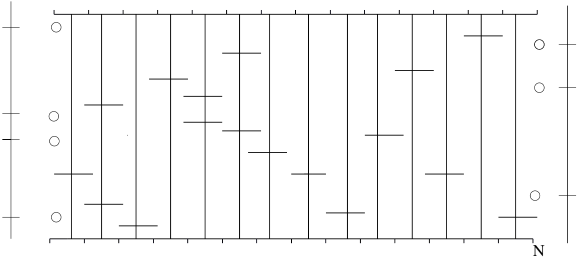
When it rings at $(N, N+1)$ put a particle at N with prob. ρ_1,

$\eta(N) = 0$ **with probability** $1-\rho_1, \quad \eta(N) = 1$ **with probability** ρ_1

and analogously if it rings at $(-1, 0)$

$\eta(0) = 0$ **with probability** $1-\rho_0, \quad \eta(N) = 1$ **with probability** ρ_0

Symmetric exclusion process: density reservoirs



Generator $L = L_0 + L'$, L_0 stirring

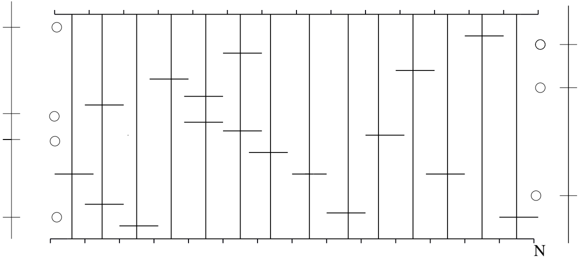
$$\begin{aligned} L'f(\eta) &= \rho_1[f(\eta^{(+,N)}) - f(\eta)] + (1-\rho_1)[f(\eta^{(-,N)}) - f(\eta)] \\ &+ \rho_0[f(\eta^{(+,0)}) - f(\eta)] + (1-\rho_0)[f(\eta^{(-,0)}) - f(\eta)] \end{aligned}$$

where $1 \geq \rho_1 > \rho_0 \geq 0$

$$\eta^{+,x}(x) = 1, \qquad \eta^{+,x}(y) = \eta(y), y \neq x$$

$$\eta^{-,x}(x) = 0, \qquad \eta^{-,x}(y) = \eta(y), y \neq x$$

Symmetric exclusion process: density reservoirs



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Symmetric exclusion process: density reservoirs

By duality:

$$\mathbb{E}\big(\eta_t(x)\big) = \sum_{y\in\Lambda_\varepsilon} p_t^0(x,y)\mathbb{E}\big(\eta_0(x)\big) + q_t(x,-1)\rho_0 + q_t(x,N+1)\rho_1$$

$p_t^0(x,y)$ is the probability a random walk goes from x to y in a time t without ever touching -1 and $N+1$

$q_t(x,-1)$ is the probability to reach -1 before $N+1$ within t.

$q_t(x,N+1)$ is the probability to reach $N+1$ before -1 within t.

Assume that the law of η_0 approximates an initial profile $\rho_0(r) \geq 0$, $r \in (0,1)$.

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Assume that the law of η_0 approximates an initial profile $\rho_0(r) \geq 0$, $r \in (0,1)$.

Symmetric exclusion process: hydrodynamic limit (density reservoirs)

Theorem

Given any $T > 0$, for all $t \leq T$

$$\lim_{\varepsilon \to 0} P\Big(\sup_r |\mathcal{M}_\ell(r, \eta_{\varepsilon^{-2}t}) - \rho(r,t)| \leq \varepsilon^a \Big) = 1$$

with $\rho(r,t)$ solution of the heat equation: $\rho(r,0) = \rho_0(r)$ and

$$\frac{\partial \rho}{\partial t} = \frac{1}{2}\frac{\partial^2 \rho}{\partial r^2}, \qquad r \in (0,1)$$

with Dirichlet b.c. $\rho(0,t) = \rho_0$, $\rho(1,t) = \rho_1$.

$$\rho(r,t) = \int G_t^0(r,z)\rho(z,0)dz + Q_t(r,0)\rho_0 + Q_t(r,1)\rho_1$$

SEP: stationary non equilibrium state, Fick's law (density reservoirs)

The unique invariant measure μ_ε is such that for any $x \in \Lambda_\varepsilon$

$$\lim_{\varepsilon \to 0, \varepsilon x \to r} \mu_\varepsilon\big(\eta(x)\big) = (\rho_1 - \rho_0) r + \rho_0$$

Microscopic current and Fick's law

$$\lim_{\varepsilon \to 0} \varepsilon^{-1} \; \mu_\varepsilon\big(\eta(x) - \eta(x+1)\big) = \rho_1 - \rho_0$$

Some of the references.

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H. Spohn (1983)
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....

Macroscopic theory: current reservoirs

Open system: $\Omega = [0,1]$, free energy $F(\rho) = \int_0^1 f(\rho(r))dr$

$$\frac{\partial \rho}{\partial t} = -\frac{\partial J}{\partial r}, \qquad J = -\kappa(\rho)\frac{\partial f'(\rho)}{\partial r} \qquad r \in (0,1)$$

Current reservoirs force a flux of mass into the system (without freezing the order parameter at the endpoints).

A current reservoir of parameter $j \in \mathbb{R}$ is such that the currents at the endpoints are:

$$J(0,t) = j\lambda\big(\rho(0,t)\big) \qquad J(1,t) = j\lambda\big(\rho(1,t)\big)$$

where $\lambda(\rho)$ is a model dependent, mobility parameter not necessarily equal to the bulk mobility $\kappa(\rho)$.
As we will see the case $\lambda \equiv 1$ corresponds to free boundary motion.

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Macroscopic theory: current reservoirs

A flux of mass $J(0,t)$ enters into the system at the point 0,
a flux of mass $J(1,t)$ leaves the system at the point 1.

Change of energy in 0 during the time interval $(t, t+dt)$ is

$$E_0 dt := f\big(\rho(0,t) + J(0,t)dt\big) - f\big(\rho(0,t)\big) \approx f'\big(\rho(0,t)\big)J(0,t)dt$$

Analogously

$$E_1 dt = f\big(\rho(1,t) - J(1,t)dt\big) - f\big(\rho(1,t)\big) \approx -f'\big(\rho(1,t)\big)J(1,t)dt$$

Thus the total change of free energy is

$$\frac{d}{dt}F^{\mathrm{tot}}(\rho(\cdot,t)) = \frac{d}{dt}\int_0^1 f(\rho(r,t))dr - E_1 - E_0$$

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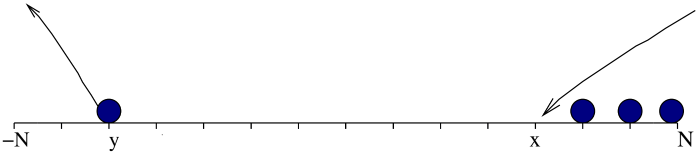
Symmetric exclusion process: current reservoirs

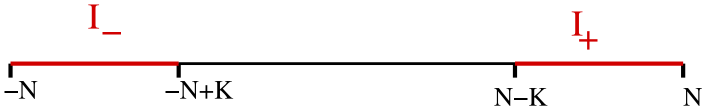
SEP in $\Lambda_\varepsilon = [-N, N] \cap \mathbb{Z}$, $N = \varepsilon^{-1}$.
Impose a macroscopic current $j > 0$ **by sending in particles from the right** at rate $\frac{j}{N}$ and **taking out particles from the left** at the same rate.



Symmetric exclusion process: current reservoirs

SEP in $\Lambda_N = [-N, N] \cap \mathbb{Z}$.

As we want the boundary processes localized at the boundaries we fix two intervals $\mathrm{I}_\pm$ of **length** K at the boundaries



we send in particles (at rate $\frac{j}{N}$) only in I_+ and take out particles only from I_-.

If I_+ is already **full** or I_- **empty**, then **our mechanisms abort.**

DM, Presutti, Tsagkarogiannis, Vares (DPTV)

Symmetric exclusion process: current reservoirs

Generator: $L = L_0 + \frac{\mathbf{j}}{\mathbf{2N}} L_b,$ L_0 stirring generator,

$L_b = L_{b,+} + L_{b,-}$ describes births and deaths near the boundaries:

$$L_{b,\pm} f(\eta) := \sum_{x \in I_\pm} D_\pm \eta(x) [f(\eta^{(x)}) - f(\eta)]$$

$$D_+ \eta(x) = (1 - \eta(x)) \eta(x+1) \cdots \eta(N)$$

$$D_- \eta(x) = \eta(x)(1 - \eta(x-1)) \cdots (1 - \eta(-N))$$

$\eta^{(x)}$ obtained from η by changing the occupation number at x.

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$\eta^{(x)}$ obtained from η by changing the occupation number at x.

Initial conditions: $\rho_0(r)$ $r \in [-1, 1]$ and the law of η_0 approximates the initial profile ρ_0.

$$\lim_{\varepsilon \to 0} P\Big(\sup_r |\mathcal{M}_\ell(r, \eta_0) - \rho_0(r)| \le \varepsilon^a \Big) = 1$$

Recall

$$\mathcal{M}_\ell(r, \eta) := \frac{1}{\ell} \sum_{x:|x-\varepsilon^{-1} r| \le \ell} \eta(x)$$

Symmetric exclusion process: hydrodynamic limit (current reservoirs)

SEP in $\Lambda_\varepsilon = [-N, N] \cap \mathbb{Z}$, $\qquad N = \varepsilon^{-1}$.

Theorem

$$\lim_{\varepsilon \to 0} P\Big(\sup_r |\mathcal{M}_\ell(r, \eta_{\varepsilon^{-2}t}) - \rho(r,t)| \le \varepsilon^a \Big) = 1$$

where

$$\frac{\partial}{\partial t}\rho(r,t) = \frac{1}{2}\frac{\partial^2}{\partial r^2}\rho(r,t), \qquad r \in (-1,1)$$

with initial datum $\rho(r,0) = \rho_0(r)$ *and* **boundary conditions** $\rho(\pm 1, t) = u_\pm(t)$ *that satisfy* **non linear coupled equations***.*

DPTV J. Stat. Phys. 2011, Electronic J. of Prob. (2011)

SEP: hydrodynamic limit (current reservoirs)

The functions $u_\pm(t)$ are the solutions of a nonlinear system of two integral equations:

$$u_\pm(t) = \int_{-1}^{1} P_t(\pm 1, r)\rho_0(r)dr + \frac{j}{2}\int_0^t \Big\{ P_s(\pm 1, 1)(\mathbf{1} - \mathbf{u}_+(\mathbf{t}-\mathbf{s})^{\mathbf{K}}) - P_s(\pm 1, -1)(\mathbf{1} - (\mathbf{1} - \mathbf{u}_-(\mathbf{t}-\mathbf{s}))^{\mathbf{K}}) \Big\} ds$$

$\mathbf{1} - \mathbf{u}_+(\mathbf{t})^{\mathbf{K}}$ is (in the limit) the probability of a hole in I_+

$\mathbf{1} - (\mathbf{1} - \mathbf{u}_-(\mathbf{t}))^{\mathbf{K}}$ the probability of a particle in I_-

$P_t(r, r')$ is the density kernel of the semigroup with generator the laplacian in $[-1, 1]$ with reflecting, Neumann, boundary conditions.

$P_t^{(N)}(x,y)$ = prob. that r.w. starting at x is at y at time $N^2 t$:

$$\begin{aligned}\mathbb{E}\big(\eta_t(x)\big) &= \sum_y P_t^{(N)}(x,y)\mathbb{E}\big(\eta_0(x)\big) \\ &+ jN\int_0^t \sum_{y\in I_\pm} P_s^{(N)}(x,y)\mathbb{E}\Big(D_\pm \eta_{t-s}(y)\Big)\end{aligned}$$

$$D_+\eta(y) = (1-\eta(y))\eta(y+1)\cdots\eta(N), \qquad y\in I_+$$
$$D_-\eta(y) = \eta(y)(1-\eta(y-1))\cdots(1-\eta(-N)), \qquad y\in I_-$$

$P_s^{(N)}(x,y) \approx \frac{1}{N}P_s(N^{-1}x,1)$ for all $y\in I_+$ and if ν= Bernoulli with parameter ρ

$$\sum_{y\in I_+}\mathbb{E}_\nu(D_+\eta(y)) = \sum_{n=1}^{K}(1-\rho)\rho^n = 0-\rho^K$$

$P_t^{(N)}(x, y)$ = prob. that r.w. starting at x is at y at time $N^2 t$:

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$$\sum_{y\in I_+} \mathbb{E}_\nu\big(D_+\eta(y)\big) = \sum_{n=1}^{K}(1-\rho)\rho^n = 0 - \rho^K$$

SEP: Fick's law (current reservoirs)

Microscopic current

$$j^{(N)}(x,t) = -\frac{1}{2}\mathbb{E}\big[\eta(x+1,t) - \eta(x,t)\big]$$

Theorem

(same assumptions)

$$\lim_{N\to\infty} Nj^{(N)}([Nr], N^2\tau) = -\frac{1}{2}\,\frac{d\rho(r,\tau)}{dr}$$

the limit currents $J_+(t)$ and $J_-(t)$ at the boundaries are:

$$J_+(t) = j\Big[1 - u_+(t)^K\Big], \quad J_-(t) = j\Big[1 - (1 - u_-(t))^K\Big]$$

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Recall that macroscopically a current reservoir of parameter $j \in \mathbb{R}$ is such that the currents at the endpoints are:

$$J(-1,t) = j\lambda\big(\rho(-1,t)\big) \qquad J(1,t) = j\lambda\big(\rho(1,t)\big)$$

where $\lambda(\rho)$ is a mobility parameter not necessarily equal to the bulk mobility $\kappa(\rho)$.

We have found in our model

$$\lambda(\rho(-1,t)) = 1 - (1 - \rho(-1,t))^K$$

$$\lambda\big(\rho(1,t)\big) = 1 - \rho(-1,t)^K$$

SEP: hydrodynamic limit: idea of proof.

Strong factorization starting from any single configuration.
DPTV: Electronic J. of Prob. (2011)

$\underline{x} = (x_1, .., x_n)$, $x_i \neq x_j$. **n body v-functions** is ($\varepsilon = N^{-1}$)

$$v_n(\underline{x}, t; \eta_0) = \mathbb{E}_{\eta_0}\Big(\prod_{i=1}^{n}\big[\eta_t(x_i) - \rho_\varepsilon(x_i, t)\big]\Big)$$

$\rho_\varepsilon(x, t)$, $x \in \Lambda_N$, $t \geq 0$ solution of the "discretized macroscopic equation" with $\rho_\varepsilon(x, 0) = \eta_0(x)$.

$$\begin{aligned} \frac{d}{dt}\rho_\varepsilon(x,t) &= \frac{1}{2}\Delta_\varepsilon\rho_\varepsilon(x,t) + \varepsilon\frac{j}{2}\Big(\mathbf{1}_{x\in I_+}D_+\rho_\varepsilon(x,t) \\ &\qquad -\mathbf{1}_{x\in I_-}D_-\rho_\varepsilon(x,t)\Big) \\ \rho_\varepsilon(x,0) &= \eta_0(x) \qquad x \in \Lambda_N \end{aligned}$$

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$\underline{x} = (x_1, .., x_n)$, $x_i \neq x_j$. **n body v-functions** is ($\varepsilon = N^{-1}$)

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$\rho_\varepsilon(x, t)$, $x \in \Lambda_N$, $t \geq 0$ solution of the "discretized macroscopic equation" with $\rho_\varepsilon(x, 0) = \eta_0(x)$.

$$\begin{aligned} \frac{d}{dt}\rho_\varepsilon(x,t) &= \frac{1}{2}\Delta_\varepsilon \rho_\varepsilon(x,t) + \varepsilon \frac{j}{2}\Big(\mathbf{1}_{x \in I_+} D_+ \rho_\varepsilon(x,t) \\ &\qquad\qquad - \mathbf{1}_{x \in I_-} D_- \rho_\varepsilon(x,t) \Big) \\ \rho_\varepsilon(x,0) &= \eta_0(x) \qquad x \in \Lambda_N \end{aligned}$$

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Theorem

$\exists \tau > 0$, $\exists \delta > 0$, *for all n there is* c_n *so that* $\forall \eta_0$ *and for all* $\mathbf{0 < t \le \tau \log N}$,

$$\sup_{\underline{x}} \left| v_n(\underline{x}; N^2 t; \eta_0) \right| \le c_n N^{-\delta n}$$

$$\frac{d}{dt}v_n(\underline{x},t) = (L_0 v_n)(\underline{x},t) + (Av)(\underline{x},t) + \frac{\mathbf{1}}{\mathbf{N}}(Bv)(\underline{x},t),$$

L_0 SEP (stirring) generator acting on $\underline{x}$

(Av) linear combination of v_{n-1} and v_{n-2} (due to the exclusion)

(Bv) linear combination of $v_{n\pm\ell}$, $\ell = 1, ..., K$ (due to the boundary process)

- terms coming from SEP analyzed in previous papers
- n body correlation feel boundary processes at rate $\frac{1}{N}T(t)$,

$T(t)$ **the local time at** $I_\pm$

- $\frac{1}{N}T(t) \approx \frac{1}{N}\sqrt{t}$, hence **small if** $t = t^* = N^{2-\beta}$**,** $\beta > 0$.

$$|\mathbf{v_n}(\underline{\mathbf{x}}, \mathbf{t}^*; \eta)| \le \mathbf{c_n N^{-\delta n}}$$

δ independent of β and η.

• Condition on the configuration at time t^* and restart, by iteration......

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SEP: stationary measure (current reservoirs)

DPTV: J. Stat. Phys. (2012)
For each N there is a unique stationary measure $\mu^{\rm st}_{\mathbf{N}}$.

Theorem

$$\lim_{N\to\infty} \mathbb{E}_{\mu^{\rm st}_N}\Big(\prod_i \big[\eta(x_i) - \varrho_{\rm st}(x_i/N)\big]\Big) = 0$$

where $\varrho_{\rm st}$ **is the unique stationary solution** *of the limit hydrodynamic equation.*

$$\varrho_{\rm st}(\mathbf{r}) = \mathbf{J}\mathbf{r} + \frac{\mathbf{1}}{\mathbf{2}}, \qquad \mathbf{J} = \mathbf{j}(\mathbf{1} - \alpha^{\mathbf{K}})$$

with α *the solution of* $\alpha(1 + j\alpha^{K-1}) = j + \frac{1}{2}$

The current in the stationary profile is $J < j$.

SEP: stationary measure (current reservoirs)

Sketch of proof. **Macroscopic profile**: $\varrho''_{\mathrm{st}} = 0$ in $(-1,1)$ + bc.

Existence: $\varrho'_{\mathrm{st}} = \frac{1}{2}\big(\varrho_{\mathrm{st}}(1) - \varrho_{\mathrm{st}}(-1)\big)$

$$\frac{1}{2}\varrho'_{\mathrm{st}} = j\big(1 - \varrho_{\mathrm{st}}(1)^K\big)$$

$$\frac{1}{2}\varrho'_{\mathrm{st}} = j\big(1 - [1 - \varrho_{\mathrm{st}}(-1)]^K\big)$$

Uniqueness: Hydrodynamic equation preserves order:
$\rho_0(r,t)$ **starts from 0** and $\rho_1(r,t)$ **from 1**
then

$$\rho_0(r,t) \le \varrho_{\mathrm{st}}(r) \le \rho_1(r,t)$$

It can be proved that

$$\boldsymbol{\rho_1(r,t) - \rho_0(r,t) \le a e^{-bt}, \quad b > 0}$$

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SEP: stationary measure (current reservoirs)

Sketch of proof. **Macroscopic profile**: $\varrho_{\rm st}'' = 0$ in $(-1, 1)$ + bc.

Existence: $\varrho_{\rm st}' = \frac{1}{2}\big(\varrho_{\rm st}(1) - \varrho_{\rm st}(-1)\big)$

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Also **the process preserves order**: it is attractive.
if $\eta_0(x) \le \xi_0(x)$, $\forall x \in \Lambda_N$ then there is a coupling such that $\eta_t(x) \le \xi_t(x)$, $\forall x \in \Lambda_N$ for all $t > 0$.

Stirring preserves order.

$$L_{b,\pm} f(\eta) := \sum_{x \in I_\pm} D_\pm \eta(x)[f(\eta^{(x)}) - f(\eta)]$$

$$D_+\eta(x) = (1-\eta(x))\eta(x+1)\cdots\eta(N)$$
$$D_-\eta(x) = \eta(x)(1-\eta(x-1))\cdots(1-\eta(-N))$$

It suffices to notice ($c(x,\eta) = D_\pm\eta(x)$)

- if $\eta(x) = \xi(x) = 0$ and $\eta \le \xi$ then $c(x,\eta) \le c(x,\xi)$
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Take $\phi \geq 0$, then

$$\mathbb{E}_{\mu^{\mathrm{st}}_{\mathbf{N}}}\left[\frac{1}{N}\sum_{x}\phi(N^{-1}x)\eta(x)\right] \leq \mathbb{E}_{\mathbf{1}}\left[\frac{1}{N}\sum_{x}\phi(N^{-1}x)\eta(x,N^2t)\right]$$

By taking $N \to \infty$,

$$\leq \int_{-1}^{1}\phi(r)\rho_1(r,t)$$

By taking $t \to \infty$,

$$\leq \int_{-\mathbf{1}}^{\mathbf{1}}\phi(\mathbf{r})\varrho_{\mathrm{st}}(\mathbf{r})$$

The reverse inequality is proved similarly.

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The reverse inequality is proved similarly.

SEP: spectral gap

Theorem

There are c and $b > 0$ independent of N so that for any initial measure ν_N

$$\|\nu_{N,t} - \mu_N^{\mathrm{st}}\| \leq cNe^{-bN^{-2}t}$$

$\|\lambda\| = \sum_\eta |\lambda(\eta)|$

DPTV: preprint (2013) http://arxiv.org/abs/1304.0624

In some respect surprising!

• With $j = 0$, $L = L_0$ (stirring process) restricted to any of the invariant subspaces $\{\eta\colon \sum \eta(x) = M\}$ has a spectral gap that scales as N^{-2}.

The full process with $L = L_0 + \dfrac{j}{N} L_b$ in a time of the same order N^2 manage to equilibrate among all the above subspaces according to μ_N^{st}.

• Density reservoirs: $L = L_0 + L'$.
Same spectral gap: $\|\nu_{N,t} - \mu_N^{\mathrm{st}}\| \leq cNe^{-bN^{-2}t}$.

Here the birth-death events are not scaled down with N.

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SEP: spectral gap

We do not have sharp information on μ_N^{st}.

We know that μ_N^{st} is close to a product measure γ_N and that the expectations $\gamma_N[\eta(x)] \sim \rho^{\mathrm{st}}(x/N)$ which does not seem detailed enough to apply the usual techniques for the spectral gap using equilibrium estimates.

Way out: use inequalities exploiting the fact that the process is attractive.

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SEP: spectral gap. Idea of proof.

For simplicity assume that $K = 2$

$$I_+ = \{N-1, N\}, \qquad I_- = \{-N, -N+1\}$$

Couple the processes starting from all 1 and from all 0.

First component is always above the second one.

$$\mathcal{X}_N = \left\{(\eta^{(1)}, \eta^{(2)}) \in (\{0,1\}\times\{0,1\})^{[-N,N]} : \eta^{(1)}(x) - \eta^{(2)}(x) \geq 0, \forall x\right\}$$

standard coupling.

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At each site x we may only have:

discrepancy $(1,0)$: $\eta_{\neq}(x) := \eta^{(1)}(x) - \eta^{(2)}(x) = 1$

full occupation $(1,1)$: $\eta_1(x) := \eta^{(1)}(x)\eta^{(2)}(x) = 1$

full void $(0,0)$: $\eta_0(x) := [1 - \eta^{(1)}(x)][1 - \eta^{(2)}(x)] = 1$

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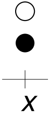

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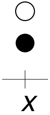

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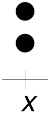

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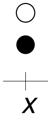

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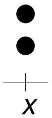

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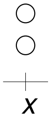


full void $(0,0)$: $\eta_0(x) := [1 - \eta^{(1)}(x)][1 - \eta^{(2)}(x)] = 1$

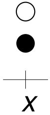
Initially $\eta_{\neq}(x) = 1$ for all x (we start with $\eta^{(1)} \equiv 1$ and $\eta^{(2)} \equiv 0$).

Theorem

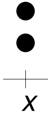
$$\sum_{x=-N}^{N} \mathbf{P}[\eta_{\neq}(x,t) = 1] \leq cNe^{-bN^{-2}t}$$

Proof: reduction to a random walk in a random moving environment.

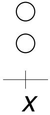
$$\xi \in \{\#, 1, 0\}^{\Lambda_N}$$



discrepancy: $\xi(x) = \#$



full occupation: $\xi(x) = 1$



full void: $\xi(x) = 0$

Evolution $\xi_t \in \{\#, 1, 0\}^{\Lambda_N}$, $t \geq 0$:

SEP (stirring) exchanges the occupation numbers of ξ.

At the boundaries $I_+ = \{N-1, N\}$, $I_- = \{-N, -N+1\}$ the changes are at rates $\frac{j}{N}$.

Three types of events: D, A and B.

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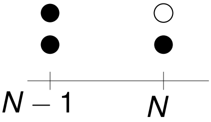
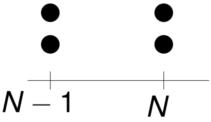
Action at the boundary $I_+ = \{N-1, N\}$

D-event at N: $\xi(N) = \#$ changes in $\xi(N) = 1$

before jump



after



or

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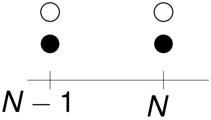
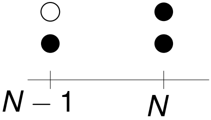


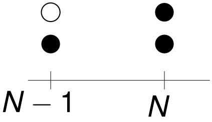
after



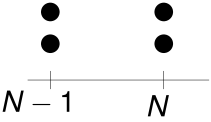
Action at the boundary $I_+ = \{N-1, N\}$

D-event at $N-1$: $\xi(N-1) = \#$ changes in $\xi(N-1) = 1$

before jump

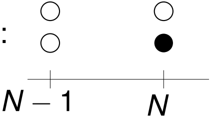


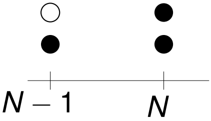
after



Action at the boundary $I_+ = \{N-1, N\}$

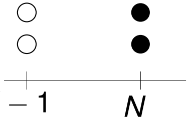
A-event: $\xi(N) = \#$ changes in $\xi(N) = 1$ and
$\xi(N-1) = 0$ changes in $\xi(N-1) = \#$

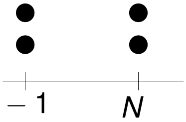
before jump: 

after 

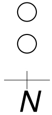
Action at the boundary $I_+ = \{N-1, N\}$

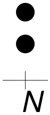
B-event at N or at $N-1$: $\xi(x) = 0$ changes in $\xi(x) = 1$, $x = N, N-1$

before jump:


after


or

before jump:


after


Call x_t the position at time t of a discrepancy.
At $t = 0$ all the sites of Λ_N are occupied by a discrepancy.

Motion of x_t: it is a random walk (stirring of the ξ-process) and:

• if an A-event occurs then it jumps from N to $N-1$ if $\xi(N-1) = 0$, and analogously from $-N$ to $-N+1$ if $\xi(-N+1) = 1$.

• if a D -event occurs then it dies (with the conditions explained before). We say that the **discrepancy goes in the state $\emptyset$.**

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We have to prove that the probability that all the discrepancies die out is exponentially close to 1.

I will prove later that it is enough to consider the case of a single discrepancy.

This lead to the analysis of a random walk x_t in a moving random environment $\xi_t \in \{\#, 0, 1\}^{[-N,N]\setminus x_t}$ when $x_t \neq \emptyset$ (i.e. it is alive)

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Random walk in a moving random environment

The process $\{x_t, t \geq 0\}$ has jump intensities at time t given by the conditional probabilities of the environment conditioned on the state of the random walk at that time.

Movements. Random walk at rate 1 to n.n. sites + extra jumps $N \to N-1$ and $-N \to -N+1$ at rates $a(\pm N, t)$.

$$a(N,t) = \frac{j}{2N} P\Big[(\xi(N-1,t)=0) \;\Big|\; x_t = N\Big]$$

Recall the A-events:

before jump:

after

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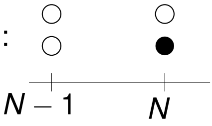

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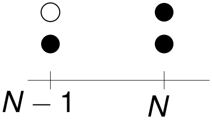

Dead rates: $d(N-1,t)$ **and** $d(N,t)$ (coming from the D-events).

before jump:

after

$$d(N-1,t) = P\Big[\xi(N,t)=1 \,\Big|\, x_t = N-1\Big]$$

ecc.....

Random walk in a moving random environment

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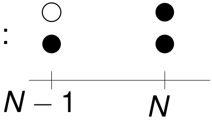
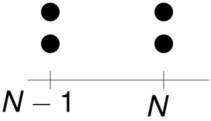
before jump: 

after 

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ecc.....

The operator $\mathcal{L}_t$

Functions f defined in $\Lambda_N \cup \{\emptyset\}$.

$$\begin{aligned} a(N,t) &= \frac{j}{2N} P\Big[(\xi(N-1,t)=0) \,\Big|\, x_t = N\Big], \\ a(-N,t) &= \frac{j}{2N} P\Big[(\xi(-N+1,t)=1 \,\Big|\, x_t = -N\Big] \end{aligned}$$

$$\begin{aligned} \mathcal{L}_t^a f(x) &= \mathcal{L}^0 f(z) + \mathbf{1}_{x=N} a(N,t)[f(N-1) - f(N)] \\ &+ \mathbf{1}_{x=-N} a(-N,t)[f(-N+1) - f(-N)], \end{aligned}$$

$\mathcal{L}_t^a$ generator of a random walk.

The full time- dependent generator $\mathcal{L}_t$ is obtained by adding the dead part.

$$\mathcal{L}_t f(x) \quad = \quad \mathcal{L}_t^a f(x) + \frac{j}{2N} d(x,t)[f(\emptyset) - f(z)],$$

$d(x,t) = 0$ if $|x| < N-1$,

$$d(N-1,t) = P\Big[\xi(N,t) = 1 \;\Big|\; x_t = N-1\Big]$$

$$d(N,t) = P\Big[\xi(N-1,t) \neq 0 \;\Big|\; x_t = N\Big]$$

Define a time-dependent Markov process $\{z_t, t \geq 0\}$ with time-dependent generator $\mathcal{L}_t$. The survival probability for this random walk is

$$\mathcal{P}\left[\mathbf{z_t} \neq \emptyset\right] = \mathbb{E}\left[\exp\{-\frac{\mathbf{j}}{\mathbf{N}}\int_0^{\mathbf{t}} \mathbf{d}(\mathbf{z_s}, \mathbf{s})\, \mathbf{ds}\}\right]$$

Lemma

Let $z_0 = x_0$, then

$$P\left[x_t \neq \emptyset\right] = \mathcal{P}\left[z_t \neq \emptyset\right]$$

Proof. It is enough to prove that for any bounded measurable function $\phi(x, \eta) = f(x)$:

$$E_{x_0,\xi_0}\left[\phi(x_t, \xi_t)\right] = \mathcal{E}_{x_0}\left[f(z_t)\right]$$

and this follows because

$$\frac{d}{dt}E_{x_0,\xi_0}\left[\phi(x_t, \xi_t)\right] = E_{x_0,\xi_0}\left[\mathcal{L}_t f(x_t)\right]$$

and also

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Random walk in a random moving environment

$$\mathcal{P}\big[z_t \neq \emptyset\big] \le \mathcal{E}\Big[\exp\{-\int_0^t d(N,s)\mathbf{1}_{z_s=N}\, ds\}\Big]$$

- There are $\delta^* > 0$ and $\kappa > 0$ so that for all $t \ge \kappa N^2$:

$$d(N,t) \ge \delta^*$$

- There are c and $b > 0$ so that calling $T^*(t)$ the total time spent at N by $z_s, 0 \le s \le t$:

$$\mathcal{E}\Big[\exp\{-j\delta^* N^{-1} T^*(t)\}\Big] \le c e^{-bN^{-2}t}, \qquad t \ge \kappa N^2$$

Random walk in a random moving environment. Many discrepancies

Initially all sites are occupied by discrepancies.

Label the initial discrepancies by assigning with uniform probability a label in $\{1, .., 2N+1\}$ to each site in $[-N, N]$

Call $(x_1, \dots, x_{2N+1})$ the sites corresponding to the labels $1, \dots, 2N+1$.

x_i is the position at time 0 of the discrepancy with label i.
At $t = 0$

$$P(x_i = x) = \frac{1}{2N+1}$$

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$$P\left[\text{there is } i : x_i(t) \neq \emptyset\right] \leq \sum_i P[x_i(t) \neq \emptyset]$$
$$= (2N+1)P[x_1(t) \neq \emptyset]$$

last equality by simmetry.

We need to estimate $P[x_1(t) \neq \emptyset]$ in an environment similar to the one before.

Random walk in a random moving environment. Many discrepancies

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We need to estimate $P[x_1(t) \neq \emptyset]$ in an environment similar to the one before.

Some open problems

• Our methods do not allow to study the large deviations of the stationary measure.

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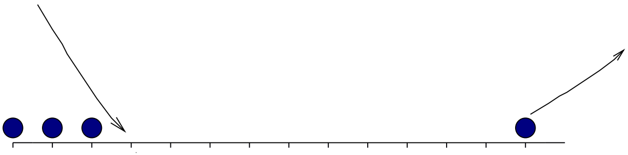

At rate εj a particle is placed at the first empty site (from the left).
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The locations of the first hole and the last particle are random.

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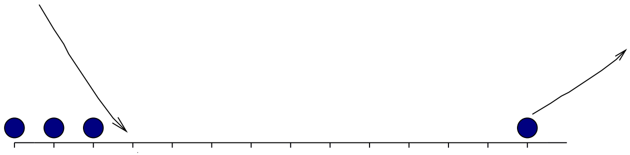

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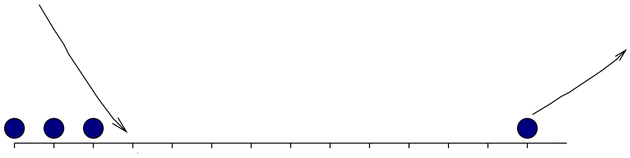

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We restrict to configurations which have a **rightmost particle** and a **leftmost hole.** The configuration space is:

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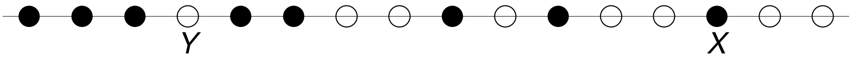

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SSEP: Free boundaries.

Markov process $\{\eta_t\}_{t\geq 0}$ with state space $\mathcal{X}$.

SSEP + birth and death at rate $j\varepsilon$, $\varepsilon > 0$ small, $j > 0$.

The generator is: $L_0 + j\varepsilon L_{\mathrm{bd}}, \quad L_{\mathrm{bd}} = L_r + L_\ell$

$$L_\ell f(\eta) := \Big(f(\eta \cup Y(\eta)) - f(\eta)\Big); \quad L_r f(\eta) := \Big(f(\eta \setminus X(\eta)) - f(\eta)\Big),$$

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Main features.

• **Topological interactions**
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• **Interaction is highly non local**: perturbation of the SEP ($L = L_0 + \varepsilon j[L_r + L_\ell]$) but $L_r + L_\ell$ are non local (need to find the last particle and the first hole). The usual techniques do not apply.

• **Hydrodynamic limits with boundary conditions on derivatives**

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Key remark.

Deaths and births happen at exponential times of parameter $j\varepsilon$ **independently of the particle configuration.**

B_t **number of rightmost particles removed** in the time $[0, t]$

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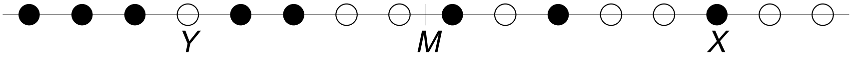
The median M is the point such:
$\#$ of holes to the left of $M = \#$ of particles to the right of M.



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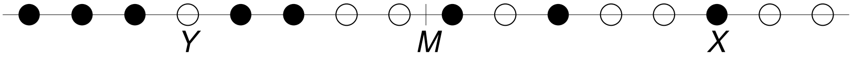

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Invariant measure

Process $\{\tilde{\eta}_t\}_{t\geq 0}$ seen from the median M:

$$\tilde{\eta}_t = \theta_{M(\eta_t)}\eta_t$$

θ = the translation $(\theta_y \eta)(x) = \eta(x-y)$

Theorem

For any $j\varepsilon > 0$ the process $\tilde{\eta}_t$ has a **unique invariant measure** $\mu_{j\varepsilon}$ *and*

$$E_{\mu_{j\varepsilon}}[X(\eta) - Y(\eta) + 1] = \frac{1}{2j\varepsilon}$$

Proof: there is a Lyapunov function. □

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Number of sites between the first hole and the last particle = $\mathbf{X}(\eta) - \mathbf{Y}(\eta) + \mathbf{1}$ **scales as** $(\mathbf{2j}\varepsilon)^{-1}$ in equilibrium.

Fick's law: the stationary current flowing in $[Y, X]$ when at the end points the densities are ρ_{left} and ρ_{right} is

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The validity of Fick's law in our case is however not obvious as the endpoints $X(\eta_t)$ and $Y(\eta_t)$ depend on time.
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Hydrodynamic limit for SEP with free boundary.

Initial condition. $\rho_0 \in C(\mathbb{R}, [0,1])$ such that there is $L_0 < 0$ and $R_0 > 0$ so that $\rho_0(r) = 0, \forall r \geq R_0$, $\rho_0(r) = 1, \forall r \leq L_0$.

Initial configuration η_0 approximates the profile ρ_0 and also

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Identification of the limit (heuristics)

Call $\mathbf{D_r}$ **the Dirac delta at** r.

$$\frac{\partial \rho}{\partial t} = \frac{1}{2}\frac{\partial^2 \rho}{\partial r^2} + jD_{L_t} - jD_{R_t}, \quad r \in [L_t, R_t]$$

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We then end up with

$$\frac{\partial\rho}{\partial t}=\frac{1}{2}\frac{\partial^2\rho}{\partial r^2},\quad r\in(L_t,R_t),$$

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Macroscopic free boundary problem

$$\frac{\partial \rho}{\partial t} = \frac{1}{2}\frac{\partial^2 \rho}{\partial r^2}, \quad r \in (L_t, R_t),$$

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It seems over-determined (too many b.c.) but is is not.

Fixed point problem: Given L_t and R_t find $\rho(r, t)$ which solves the heat equation with b.c. (1).

Determine L_t and R_t so that the spatial derivative of ρ at these points are equal to $-2j$.

Macroscopic free boundary problem

$$\frac{\partial \rho}{\partial t} = \frac{1}{2}\frac{\partial^2 \rho}{\partial r^2}, \quad r \in (L_t, R_t),$$
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Reduction to the classical Stefan problem.

By "differentiating" $\rho(L_t, t) = 1$ and $\rho(R_t, t) = 0$ we get

$$\frac{dL_t}{dt} = \frac{1}{4j}\frac{\partial^2 \rho}{\partial r^2}(L_t, t), \qquad \frac{dR_t}{dt} = \frac{1}{4j}\frac{\partial^2 \rho}{\partial r^2}(R_t, t)$$

Define $u = \rho_r$

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial r^2}, \quad r \in (L_t, R_t), \qquad u(L_t, t) = -2j = u(R_t, t)$$

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This is now the classical Stefan problem: a diffusive equation with Dirichlet b.c., on an interval whose endpoints evolve with velocity determined by the derivative of the solution. To recover ρ we set

$$\rho(r, t) := -\int_r^{R_t} u(r', t)\, dr', \quad r \in [L_t, R_t]$$

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Some references.

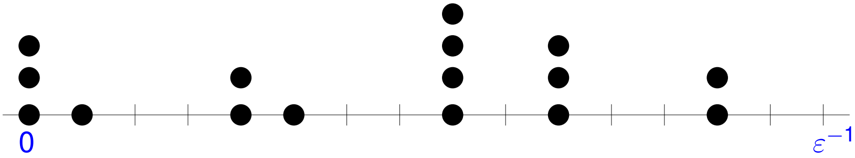
- Hubert Lacoin (2013) *The scaling limit of polymer pinning dynamics and a one dimensional Stefan freezing problem*.

- Claudio Landim; Glauco Valle. (2006) *A microscopic model for Stefan's melting and freezing problem*.

Independent particles with births and deaths.

(G. Carinci, C. Giardina, DM, E. Presutti)

Independent random walks in $[0, \varepsilon^{-1}] \cap \mathbb{Z}$ (jumps outside suppressed).



At rate $j\varepsilon$ a new particle is created at 0,

At rate $j\varepsilon$ the rightmost particle is deleted

Generator

$\varepsilon^{-1} \in \mathbb{N}$, $[0, \varepsilon^{-1}] \equiv [0, \varepsilon^{-1}] \cap \mathbb{Z}$ $\qquad \xi \in [0, \varepsilon^{-1}]^{\mathbb{N}}$

$$\xi(x) = \text{number of particles at } x, \qquad x \in [0, \varepsilon^{-1}]$$

Generator: $L = L_0 + j\varepsilon[L_a + L_d]$: L_0= generator of the independent symmetric random walks

L_a = add a particle at the origin

$$L_a f(\xi) = f(\xi + \mathbf{1}_0) - f(\xi)$$

L_d = remove a particle at the rightmost occupied site

$$L_b f(\xi) = f(\xi - \mathbf{1}_X) - f(\xi)$$

$$X : \xi(X) > 0, \qquad \xi(y) = 0 \qquad \forall y > X$$

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Hydrodynamic limit.

Theorem

$\exists\ \rho_t = \rho_t(r), r \in [0,1], t \geq 0$, *non negative and in* L^1 *such that "*$\xi_{\varepsilon^{-2}t}$ *converges to* ρ_t *weakly" which means:*

$$\lim_{\varepsilon\to 0} P_\xi^{(\varepsilon)}\Big[\max_{x\in[0,\varepsilon^{-1}]} |\varepsilon F_\varepsilon(x;\xi_{\varepsilon^{-2}t}) - F(\varepsilon x;\rho_t)| > \zeta\Big] = 0$$

for any $\zeta > 0$.

$$F_\varepsilon(x;\xi) := \sum_{y=x}^{\varepsilon^{-1}} \xi(y); \quad F(r;\rho) := \int_r^1 \rho(r')dr'$$

proved in [CDGP] under suitable assumptions on the initial datum.

Strategy: inequalities

When a particle dies it is retained becoming a "ghost".
Ghost and true particles together are independent random walks

We thus know well the overall configuration at a time $T = \varepsilon^{-2}\delta$; to get the true particles configuration we must "guess" which are the ghosts and delete them.

N, the random number of ghosts at time T, is an independent Poisson variable of mean jT.

Natural candidates for the ghosts:
(i) the N rightmost particles at time T
(ii) the N particles at time T which were the rightmost particles at time 0.

They are both incorrect yet are lower and upper bounds (in a suitable topology) which become accurate as first $\varepsilon \to 0$ and then $\delta \to 0$.

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Identification of the limit (heuristics)

If $j = 0$ (i.e. no births and deaths):

$$\frac{\partial \rho}{\partial t} = \frac{1}{2}\frac{\partial^2 \rho}{\partial r^2}, \qquad \left.\frac{\partial \rho}{\partial r}\right|_0 = \left.\frac{\partial \rho}{\partial r}\right|_1 = 0$$

The heat equation with Neumann boundary conditions.

Adding births and deaths:

$$\frac{\partial \rho}{\partial t} = \frac{1}{2}\frac{\partial^2 \rho}{\partial r^2} + jD_0 - jD_{R_t}, \quad r \in [0, R_t]$$

where D_r is the Dirac delta at r.

R_t the smallest point such that $\rho(r,t) = 0$ for $r > R_t$
(supposing $R_0 < 1$ and t small).

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Replace Neumann condition by symmetry under reflection around 0:

$$\frac{\partial \rho}{\partial t} = \frac{1}{2}\frac{\partial^2 \rho}{\partial r^2} + 2jD_0 - jD_{R_t} - jD_{-R_t}, \quad r \in [-R_t, R_t],$$

- $\rho(r,t) = \rho(-r,t)$
- $\rho(R_t, t) = 0$
- $\rho(r, 0) = \rho_{\mathrm{init}}(r)$

For any test function $\phi(r,t)$:

$$-\int \phi_t \rho = \int \frac{1}{2}\phi_{rr}\rho + j\Big(2\phi(0,t) - \phi(R(t),t) - \phi(-R(t),t)\Big)$$

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Classical solutions. If there is a solution $\rho(r,t)$ which is smooth in $(0, R_t)$, then (integrating by parts)

$$\frac{\partial \rho}{\partial t} = \frac{1}{2}\frac{\partial^2 \rho}{\partial r^2}, \quad \rho(R_t, t) = 0, \; \frac{\partial \rho}{\partial r}|_{r=0^+} = \frac{\partial \rho}{\partial r}|_{r=R_t^-} = -2j$$

Fixed point problem: Given R_t we find $\rho(r,t)$ which solves the heat equation with 0 boundary conditions at $\pm R_t$.

Determine R_t so that the derivative at R_t is equal to $-2j$.

Stefan problem

Existence follows by reducing to the classical Stefan problem. By differentiating $\rho(R_t, t) = 0$:

$$\dot{R}_t = j^{-1} \frac{1}{2} \frac{\partial^2 \rho}{\partial r^2}\Big|_{r=R_t^-}$$

Define: $u := \frac{\partial \rho}{\partial r}$, then

$$\frac{\partial u}{\partial t} = \frac{1}{2} \frac{\partial^2 u}{\partial r^2}, \quad u(0,t) = u(R_t, t) = -j,$$

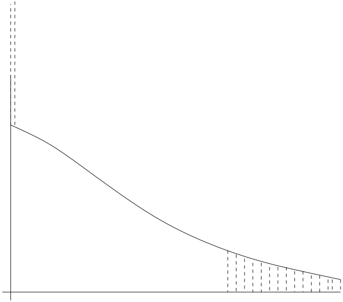
with $u(r,0) = \frac{\partial \rho_{\mathrm{init}}}{\partial r}$ and

$$\dot{R}_t = -j^{-1} \frac{1}{2} \frac{\partial u}{\partial r}\Big|_{r=R_t^-}$$

One can then check that $\rho(r,t) := -\int_r^{R_t} u(r,t)$ solves the original problem.

Weak solutions, via barriers

$K^{(\delta)}\rho$ "the cut and paste map" acting on ρ:



The mass in shaded areas are = $j\delta$, mass on the right is moved to the origin

$$\mathbf{K^{(\delta)}u = j\delta D_0 + u\,1_{r\in[0,R_\delta(u)]}}$$

$R_\delta(u)$ such that $\displaystyle\int_{R_\delta}^1 u(r)dr = j\delta.$

The barriers.

$G_\delta^{\text{neum}}(r, r')$ = Green function of the heat equation in $[0, 1]$ with Neumann boundary conditions:

$$G_t^{\text{neum}}(r, r') = \sum_k G_t(r, r'_k), \quad G_t(r, r') = \frac{e^{-(r-r')^2/2t}}{\sqrt{2\pi t}}$$

r'_k being the images of r' under repeated reflections of the interval $[0, 1]$.

$$\mathbf{S}_{\mathbf{n}\delta}^{(\delta,-)}(\rho) := K^{(\delta)} G_\delta^{\text{neum}} \cdots K^{(\delta)} G_\delta^{\text{neum}} \rho \qquad (n \text{ times})$$

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Mass trasport inequalities.

Call $F(r;u) := \int_r^1 u(r)dr, \quad u \geq 0$

Definition

$$u \leq v \quad \text{iff} \quad F(r;u) \leq F(r;v), \qquad \forall r \in [0,1]$$

$F(r;u)$ is a non increasing function of r which starts at 0 from the total mass of u: $F(0;u) = \int_0^1 u(r)dr$.

The graph of $F(r;u)$ is *"the interface of u"* and $u \leq v$ means that the interface of v is not below the interface of u.

$$\mathcal{U} = \{u = cD_0 + \rho, c \geq 0, \rho \in L^\infty([0,1], \mathbb{R}_+)\}$$

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Mass trasport inequalities.

Lemma

For any $\delta > 0$ and any integer n

$$S_{n\delta}^{(\delta,-)}(\rho) \leq S_{n\delta}^{(\delta,+)}(\rho)$$

(it is better to do the cut and paste earlier)

Actually we prove that for all δ, δ' and t such that $t = k\delta = k'\delta'$:

$$S_t^{(\delta,-)}(u) \leq S_t^{(\delta',+)}(u)$$

Weak solution via barriers.

Definition. ρ_t is a weak solution in the sense of barriers if $\rho_0 = u$ and for any δ and n:

$$S_{n\delta}^{(\delta,-)}(u) \leq \rho_{n\delta} \leq S_{n\delta}^{(\delta,+)}(u)$$

Theorem

Under suitable assumption on ρ_{init} there is a unique weak solution ρ_t (in the sense of barriers) with $\rho_0 = \rho_{\mathrm{init}}$.

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Hydrodynamic limit.

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Under suitable assumption on ρ_{init} the hydro-limit ρ_t of $\xi_{\varepsilon^{-2}t}$ is the unique weak solution (in the sense of barriers).

(precise statement later)

Theorem 2. Classical solutions are weak solutions.

Work in progress. Different strategies: P.Ferrari (use approximation via harmonic lattice maps), S. Olla (control the limit of $S^{\delta,\pm}$ via expansion in δ) CGDP (use again inequalities proving that the classical solution is the hydro-limit of a particle system that is in between the barriers)

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Proofs.

$\xi_t^{(\delta,-)}$ is defined as :

- independent random walks for $t < \varepsilon^{-2}\delta$;

at $t = \varepsilon^{-2}\delta$

- cut the N_- rightmost particles and add N_+ particles at 0:

$\mathbf{N}_\pm$ being the **number of particles created and deleted** in the true process ξ_t for $t \in [0, \varepsilon^{-2}\delta]$.

By iteration it is defined for all $t = n\delta$.

$\xi_t^{(\delta,+)}$ is defined with same procedure but anticipating the cut and paste.
namely at time 0^+ we kill the n^- rightmost particles and create n^+ new particles at 0 and then let evolve independently till $t \leq \varepsilon^{-2}\delta$, then iterate....

Proofs.

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- independent random walks for $t < \varepsilon^{-2}\delta$;

at $t = \varepsilon^{-2}\delta$

- cut the N_- rightmost particles and add N_+ particles at 0:

$\mathbf{N}_\pm$ being the **number of particles created and deleted** in the true process ξ_t for $t \in [0, \varepsilon^{-2}\delta]$.

By iteration it is defined for all $t = n\delta$.

$\xi_t^{(\delta,+)}$ is defined with same procedure but anticipating the cut and paste.
namely at time 0^+ we kill the n^- rightmost particles and create n^+ new particles at 0 and then let evolve independently till $t \leq \varepsilon^{-2}\delta$, then iterate....

Proofs.

$\xi_t^{(\delta,-)}$ is defined as :

- independent random walks for $t < \varepsilon^{-2}\delta$;

at $t = \varepsilon^{-2}\delta$

- cut the N_- rightmost particles and add N_+ particles at 0:

$\mathbf{N}_\pm$ being the **number of particles created and deleted** in the true process ξ_t for $t \in [0, \varepsilon^{-2}\delta]$.

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The total number of particles

To define $N_{\pm}$ we use the random variable

$$|\xi_t| = \text{total number of particles at time } t$$

$|\xi_t|$ has the law of a random walk on $\mathbb{N}$ which jumps with equal probability by ± 1 after an exponential time of parameter $j\varepsilon$, the jumps leading to -1 being suppressed.

$$N_{k,+} = \# \quad \textbf{upwards} \text{ jumps of } |\xi_s| \text{ for } s \in [k\varepsilon^{-2}\delta, (k+1)\varepsilon^{-2}\delta]$$

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$N^0_{k,+}$, $N^0_{k,-}$ independent Poisson variables with average $\varepsilon^{-1} j\delta$.

$$N_{k,+} = N^0_{k,+}, \qquad N_{k,-} \le N^0_{k,-}$$

because if the independent clock rings at a time s and $|\xi_s| = 0$, then at s there is no jump.

Definition (Assumptions on the initial particle configuration)

$$\max_{x \in [0,\varepsilon^{-1}]} \Big| \mathcal{A}_\ell(x,\xi) - \mathcal{A}'_\ell(x,\rho_{\rm init}) \Big| \le \varepsilon^a$$

$\rho_{\rm init} \in C([0,1], \mathbb{R}_+)$, $\rho_{\rm init}(r) = 0, r \in [R_0, 1]$

$$|\varepsilon R(\xi) - R_0| \le \varepsilon^a$$

$$\mathcal{A}_\ell(x,\xi) := \frac{1}{\ell} \sum_{y=x}^{x+\ell-1} \xi(y), \quad \mathcal{A}'_\ell(x,\rho) = \frac{1}{\varepsilon\ell} \int_{\varepsilon x}^{\varepsilon(x+\ell)} \rho(r) dr$$

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$$|\xi_0| \geq \varepsilon^{-1} \int_0^1 \rho_{\rm init}(r) dr - \varepsilon^{-1+a} \geq \varepsilon^{-1} C, \qquad C > 0$$

Lemma

Given $T > 0$ and $\gamma > 0$ define

$$\mathcal{G} = \left\{ |N^0_{k,+} - \varepsilon^{-1} j\delta| \leq \varepsilon^{-\frac{1}{2}-\gamma}; |N^0_{k,-} - \varepsilon^{-1} j\delta| \leq \varepsilon^{-\frac{1}{2}-\gamma}, \forall k \leq \delta^{-1} T \right\}$$

In the good set $\mathcal{G}$, for all $k \leq \delta^{-1} T$

$$N_{k,+} = N^0_{k,+}, \qquad N_{k,-} = N^0_{k,-}$$

and

$$P[\mathcal{G}] \geq 1 - c_n \varepsilon^n$$

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Inequalities.

Definition

- $\xi \leq \xi'$ iff $F_\varepsilon(x;\xi) \leq F_\varepsilon(x;\xi')$ for all $x \in [0, \varepsilon^{-1}]$

$$F_\varepsilon(x;\xi) := \sum_{y \geq x} \xi(y)$$

- The process $(\xi_t)_{t\geq 0}$ is stochastically $\leq$ than the process $(\xi'_t)_{t\geq 0}$ if they can be realized on a same probability space where the inequality holds pointwise (almost surely).

Theorem

$$\xi^{(\delta,-)}_{k\varepsilon^{-2}\delta} \leq \xi_{k\varepsilon^{-2}\delta} \leq \xi^{(\delta,+)}_{k\varepsilon^{-2}\delta}, \qquad \textit{for all } k$$

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Hydrodynamic limit for the approximating processes

$$\mathcal{A}_\ell(x;\xi) := \frac{1}{\ell}\sum_{y=x}^{x+\ell-1}\xi(y), \qquad \ell = \varepsilon^{-b}, b \in (0,1)$$

$\langle \xi_t \rangle$= expectation

Theorem

Let b be suitably close to 1and $T > 0$. Then for any $\zeta > 0$ and and n : $n\delta \le T$,

$$\lim_{\varepsilon\to 0} P_\xi^{(\varepsilon)}\Big[\max_{x\in[0,\varepsilon^{-1}-\ell]} |\mathcal{A}_\ell(x;\xi^{(\delta,\pm)}_{n\varepsilon^{-2}\delta}) - \mathcal{A}_\ell(x;\langle \xi^{(\delta,\pm)}_{n\varepsilon^{-2}\delta}\rangle)| > \zeta\Big] = 0$$

$$\lim_{\varepsilon\to 0} \langle \xi^{(\delta,\pm)}_{n\varepsilon^{-2}\delta}\rangle = S^{(\delta,\pm)}_{n\delta}(\rho)$$

Hydrodynamic limit for the process

previous Theorem and

$$\xi^{(\delta,-)}_{n\varepsilon^{-2}\delta} \le \xi_{n\varepsilon^{-2}\delta} \le \xi^{(\delta,+)}_{n\varepsilon^{-2}\delta}, \quad \text{and} \quad \lim_{\varepsilon\to 0} \langle \xi^{(\delta,\pm)}_{n\varepsilon^{-2}\delta} \rangle = S^{(\delta,\pm)}_{n\delta}(\rho_{\mathrm{init}})$$

imply that

$$S^{(\delta,-)}_{n\delta}(\rho_{\mathrm{init}}) \le S^{(\delta,+)}_{n\delta}(\rho_{\mathrm{init}})$$

Theorem

There is a unique element ρ_t separating the barriers:

$$S^{(\delta,-)}_{n\delta}(\rho_{\mathrm{init}}) \le \rho_t \le S^{(\delta,+)}_{n\delta}(\rho_{\mathrm{init}})$$

Such an element is equal to the hydrodynamic limit of $\{\xi_t\}$.

$$\lim_{\varepsilon\to 0} P_\xi \left[\max_{x\in[0,\varepsilon^{-1}]} |\varepsilon F_\varepsilon(x;\xi_{\varepsilon^{-2}t}) - F(\varepsilon x;\rho_t)| \le \zeta \right] = 1$$

Proof (sketch).

• **Monotonicity**: as functions of δ, $S_{n\delta}^{(\delta,-)}(\rho)$ is non decreasing and $S_{n\delta}^{(\delta,+)}(\rho)$ is non increasing: for all $\delta = k\delta'$,

$$S_{m\delta}^{(\delta,-)}(\rho) \le S_{m\delta}^{(\delta',-)}(\rho), \qquad S_{m\delta}^{(\delta',+)}(\rho) \le S_{m\delta}^{(\delta,+)}(\rho)$$

• **Regularity**. $S_t^{(\delta,+)}(\rho)$, $t \in \delta\mathbb{N}$ is space-time equicontinuous.

• **Closeness**. For all $t > 0$

$$|S_t^{(\delta,+)}(u) - S_t^{(\delta,-)}(u)|_1 \le 4j\delta, \quad \text{for all } t > 0 \text{ in } \delta\mathbb{N}.$$

$|\cdot|_1$ is the total variation norm

(It follows from: $|K^{(\delta)}u - K^{(\delta)}v|_1 \le |u - v|_1$; $|K^{(\delta)}u - u| \le 2j\delta$, $|G_\delta^{\text{neum}}u - G_\delta^{\text{neum}}v|_1 \le |u - v|_1$).

Stationary macroscopic profiles.

$$0 = \frac{1}{2}\frac{\partial^2 \rho}{\partial r^2}, \quad \rho(R) = 0, \quad \frac{\partial \rho}{\partial r}|_{r=0^+} = \frac{\partial \rho}{\partial r}|_{r=R^-} = -2j$$

$$\int_0^1 \rho = M$$

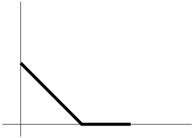
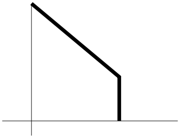


Figure : Stationary solution for $M < j$, $r \in [0, 1]$



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Manifold of stationary macroscopic profiles.

$\mathcal{M} := \{\rho^{(M)}, M > 0\}$ one-dimensional manifold of classical stationary solutions.

$$\rho^{(M)}(r) = \begin{cases} 2j(R-r), & R \le 1 \\ 0 & r > R \end{cases}, \qquad \int_0^1 \rho^{(M)}(r)dr = M \le j$$

$$\rho^{(M)}(r) = 2j(1-r) + \rho^{(M)}(1), \qquad \int \rho^{(M)} = M > j$$

Stability of the manifold of stationary profiles.

Theorem

Let $\int_0^1 \rho_{\text{init}}(r)dr = M$ *and* ρ_t *the hydro-limit starting from* ρ_{init}*.*
Then, as $t \to \infty$*,* ρ_t *converges weakly to* $\rho^{(M)}$ *in the sense that*

$$\lim_{t\to\infty} F(r;\rho_t) = F(r;\rho^{(M)}), \qquad \forall r \in [0,1]$$

$$F(r;u) = \int_r^1 u(r)dr$$

Proof : loss of memory.

Two initial configurations ξ and $\tilde{\xi}$ with $|\xi| = |\tilde{\xi}| = n$
ξ approximates ρ_{init} and $\tilde{\xi}$ approximates $\rho^{(M)}$

Coupling of the two processes $\{\xi_t\}$ and $\{\tilde{\xi}_t\}$,

• For the free evolution we label the particles and consider n independent random walks starting from $\underline{x}$ and n independent r.w. starting from $\underline{y}$ with the rule that when particles with same label meet they stick together then after.

• Births are easy since the position of the born particles is the origin for both processes and so they stick together forever.

• For the deaths there is a way to relabel the particles so that the distance between the position of particles with same label decreases.

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Super-hydrodynamic limit

Hydrodynamic limit: $\xi_{\varepsilon^{-2}t} \to \rho_t$ in the limit $\varepsilon \to 0$ keeping t fixed.

$\rho_t \to \rho^{(M)}$ in the limit $t \to \infty$

Interchange of limits in not allowed!.
There is a second time scale.

Total number $|\xi_t|$ of particles at time t performs a symmetric random walk with jumps by ± 1 at rate εj.

The density $\varepsilon|\xi_t|$ changes after times of the order ε^{-3}.

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Brownian motion on the manifold of stationary profiles

Theorem

Let $M_t^\varepsilon := \varepsilon|\xi_{\varepsilon^{-3}t}|$ *then for any* $r \in [0,1]$ *and any* $t > 0$*,*

$$\lim_{\varepsilon \to 0} P_\xi^{(\varepsilon)} \left[\sup_{r \in [0,1]} |\varepsilon F_\varepsilon(r; \xi_{\varepsilon^{-3}t}) - F(r; \rho^{(M_t^\varepsilon)})| \leq \zeta \right] = 1$$

Moreover M_t^ε *converges in law as* $\varepsilon \to 0$ *to a brownian motion on* $\mathbb{R}_+$ *with reflecting boundary conditions at* 0*.*