Voronoi diagram on a Riemannian manifold

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Motivation

Aim : Show a link between mean characteristics of the Voronoi cells and local characteristics of the manifold

Framework

- M compact Riemannian manifold, with its Riemannian metric d,
- dx volume element induced by the metric,
- Φ Poisson point process of intensity λdx and $x_0 \in M$ added to Φ,
- The Voronoi cell of x_0 defined by

$$C(x_0, \Phi) = \{y \in M, d(x_0, y) \leq d(x, y), \forall x \in \Phi\}$$

- N the number of vertices.

Outline

Mean number of vertices on the sphere

wlog, assume x_0 to be the North pole on the sphere of constant curvature K (of radius $\frac{1}{\sqrt{K}}$)

$$\mathbb{E}[N(\mathcal{C})] = 6 - \frac{3K}{\pi\lambda} + e^{-\frac{4\pi\lambda}{K}} \left(\frac{3K}{\pi\lambda} + 6 \right)$$

Miles (1971) : n uniform points on the sphere

Strategy for a general surface

Get this kind of formula for an arbitrary surface

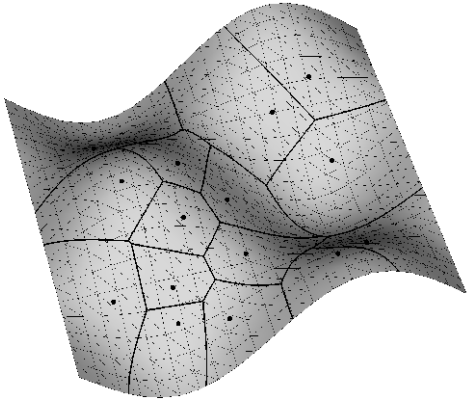
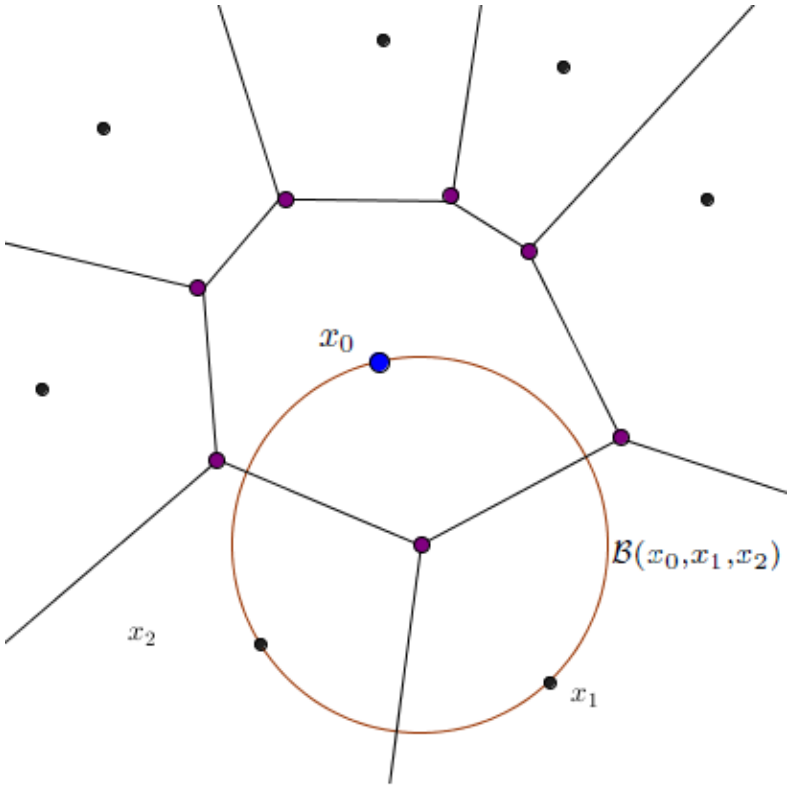


image:R.Kunze

- **Step 1:** characterize vertices of $\mathcal{C}$
- **Step 2:** apply Mecke-Slivnyak formula
- **Step 3:** use geodesic polar coordinates
- **Step 4:** make a Blaschke-Petkantschin type change of variables
- **Step 5:** find the volume of a geodesic ball

Sketch of proof



Step 1: characterize vertices of $\mathcal{C}$

Sketch of proof

$$\mathbb{E}[N(\mathcal{C})] = \mathbb{E}\left[\sum_{x_1,x_2\in\Phi}\ \sum_{\text{circumscribed balls}} \mathbb{1}_{\{\mathcal{B}(x_0,x_1,x_2)\cap\Phi=\emptyset\}}\right]$$

Step 1: characterize vertices of $\mathcal{C}$

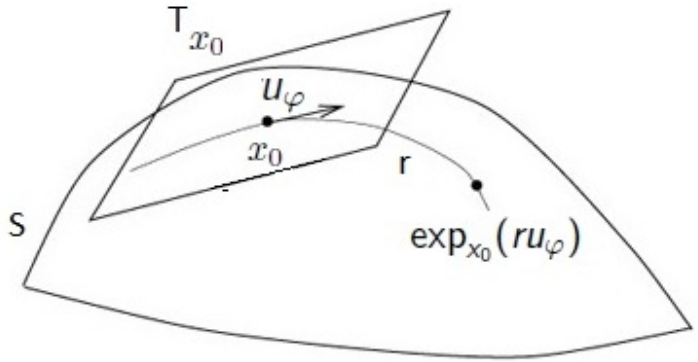
Sketch of proof

$$\mathbb{E}[N(\mathcal{C})] = \frac{\lambda^2}{2} \iint_{x_1, x_2 \in M} \sum_{\text{circumscribed balls}} e^{-\lambda \operatorname{vol}(\mathcal{B}(x_0, x_1, x_2))} dx_1 dx_2$$

1. Points "far" from x_0 contribute negligibly.
2. For points around x_0, only one circumscribed ball contributes.

Step 2: apply Mecke Slivnyak formula

Exponential map



Around x_0, M can always be parametrized by its geodesic polar coordinates (r, φ), ie

$$x = \exp_{x_0}(ru_\varphi)$$

Step 3: use geodesic polar coordinates

Rauch theorem

$$dx = f(r, \varphi) dr d\varphi$$

Let K denote the Gaussian curvature.

Rauch theorem (1951)

Si $0 < \delta \leq K \leq \Delta$

$$\frac{\sin(\sqrt{\Delta} r)}{\sqrt{\Delta}} \leq f(r, \varphi) \leq \frac{\sin(\sqrt{\delta} r)}{\sqrt{\delta}}$$

Application: $\delta = K(x_0) - \varepsilon$, $\Delta = K(x_0) + \varepsilon$

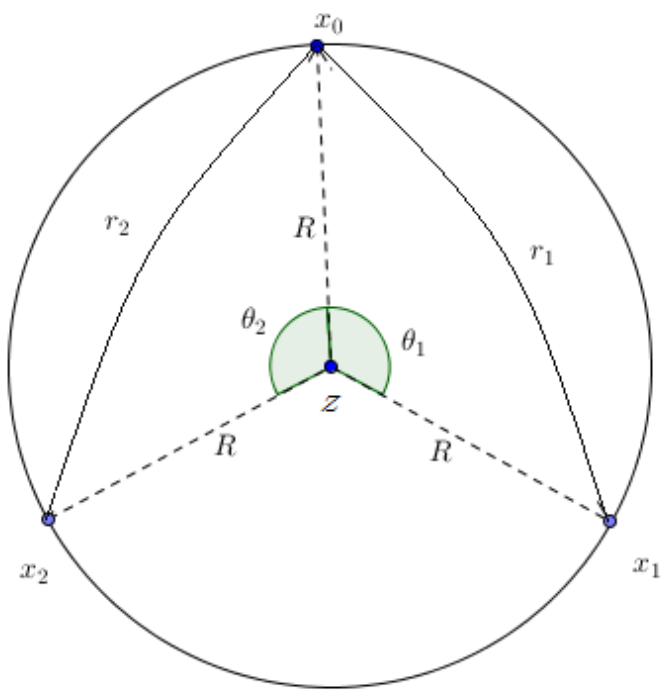
Step 3: use geodesic polar coordinates

Sketch of proof

$$E[N(\mathcal{C})] = \frac{\lambda^2}{2} \int_{\substack{(r_1,\varphi_1)\\(r_2,\varphi_2)}} e^{-\lambda \operatorname{vol}(\mathcal{B}(x_0,x_1,x_2))} \times \left(r_1 - \frac{K(x_0)r_1^3}{6} + o(r_1^3)\right)\left(r_2 - \frac{K(x_0)r_2^3}{6} + o(r_2^3)\right) dr_1 d\varphi_1 dr_2 d\varphi_2 + O(e^{-c\lambda})$$

Step 3: use geodesic polar coordinates

Sketch of proof



$r_1 = ?$

$r_2 = ?$

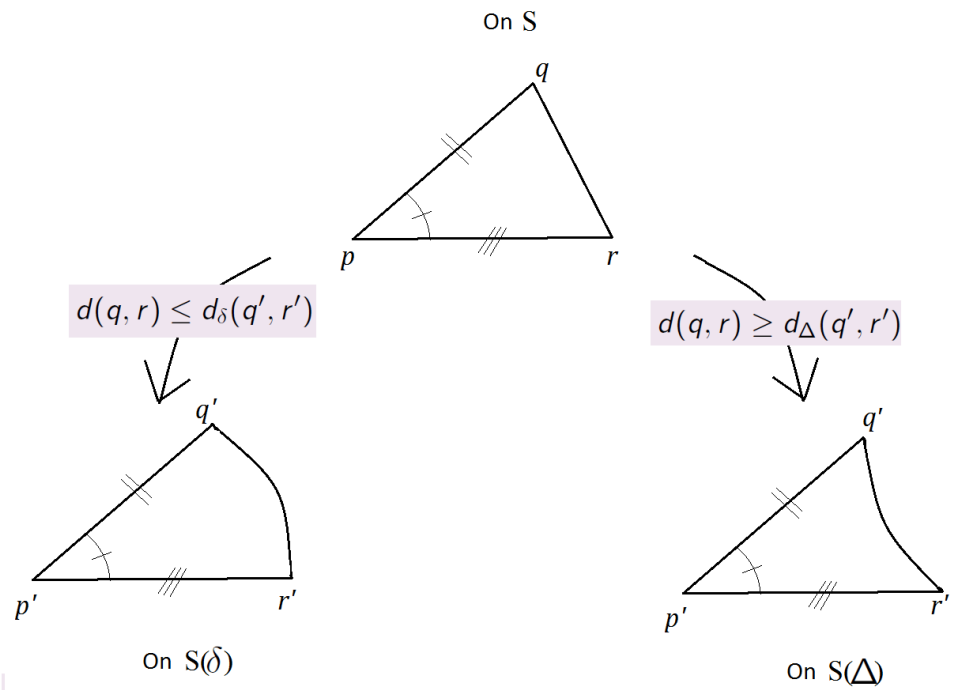
$\varphi_1 = ?$

$\varphi_2 = ?$

Step 4: make a Blaschke-Petkantschin type change of variables

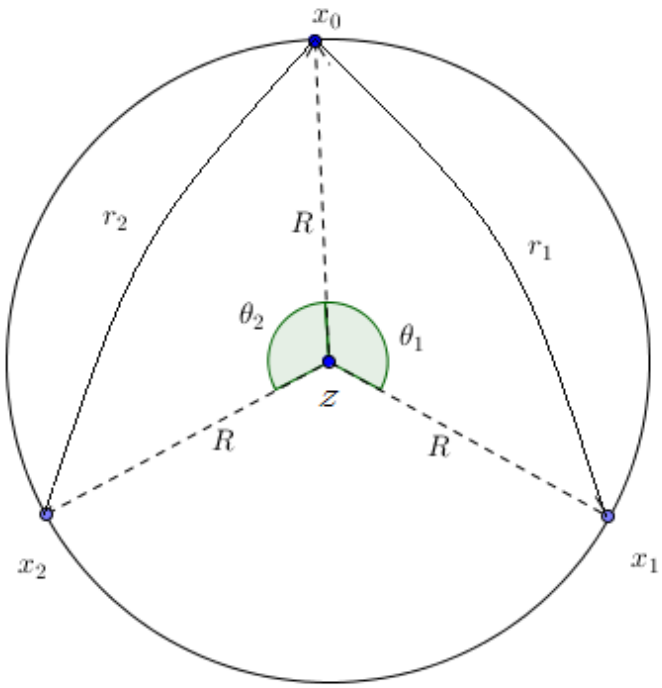
Toponogov theorem

If $\delta \leq K \leq \Delta$



Step 4: make a Blaschke-Petkantschin type change of variables

Sketch of proof



$$r_1 = 2\sin(\theta_1/2)R - \frac{K(x_0)R^3}{3}\sin(\theta_1/2)\cos^2(\theta_1/2) + o(R^3)$$

$$r_2 = 2\sin(\theta_2/2)R - \frac{K(x_0)R^3}{3}\sin(\theta_2/2)\cos^2(\theta_2/2) + o(R^3)$$

$$\varphi_1 = \varphi + \frac{\pi}{2} - \frac{\theta_1}{2} + \frac{K(x_0)R^2}{4}\sin(\theta_1) + o(R^2)$$

$$\varphi_2 = \varphi + \frac{\pi}{2} - \frac{\theta_2}{2} + \frac{K(x_0)R^2}{4}\sin(\theta_2) + o(R^2)$$

Step 4: make a Blaschke-Petkantschin type change of variables

Sketch of proof

$$\mathbb{E}[N(\mathcal{C})] = 2\lambda^2 I \int_{\varphi} \int_{R} e^{-\lambda \operatorname{vol}(\mathcal{B}(z,R))} \left(R^3 - \frac{K(x_0)R^5}{2} + o(R^5) \right) dR d\varphi + O(e^{-c\lambda})$$

where

$$I = \int_{\theta_1,\theta_2} \sin\left(\frac{\theta_1}{2}\right) \sin\left(\frac{\theta_2}{2}\right) \left| \sin\left(\frac{\theta_1 - \theta_2}{2}\right) \right| d\theta_1 d\theta_2$$

Step 4: make a Blaschke-Petkantschin type change of variables

Volume of small geodesic balls

Bertrand-Diquet-Puiseux theorem (1848)

When $R \to 0$, $x \in S$

$$\mathrm{vol}(\mathcal{B}(z,R)) = \pi R^2 - \frac{K(z)\pi}{12} R^4 + o(R^4)$$

Step 5: find the volume of the circumscribed ball

Result

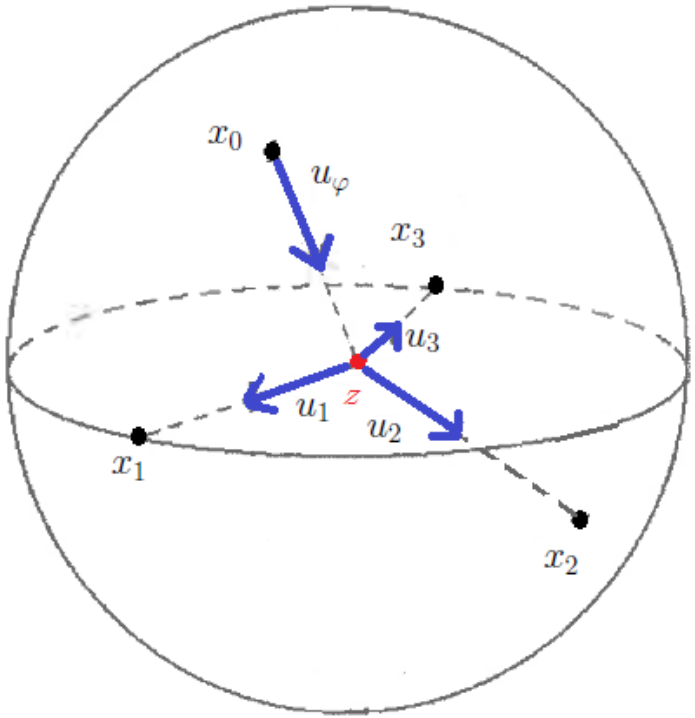
$$\mathbb{E}[N(\mathcal{C})] = 12\pi^2\lambda^2 \int_0^{R_{max}} e^{-\lambda(\pi R^2 - \frac{\pi K(x_0)R^4}{12} + o(R^4))} \times [R^3 - \frac{K(x_0)R^5}{2} + o(R^5)]dR + O(e^{-c\lambda})$$

When λ goes to infinity, Laplace's method yields

Mean number of vertices

$$\mathbb{E}[N(\mathcal{C})] = 6 - \frac{3K(x_0)}{\pi\lambda} + o\left(\frac{1}{\lambda}\right)$$

The dimension n



The vertices of $\mathcal{C}$

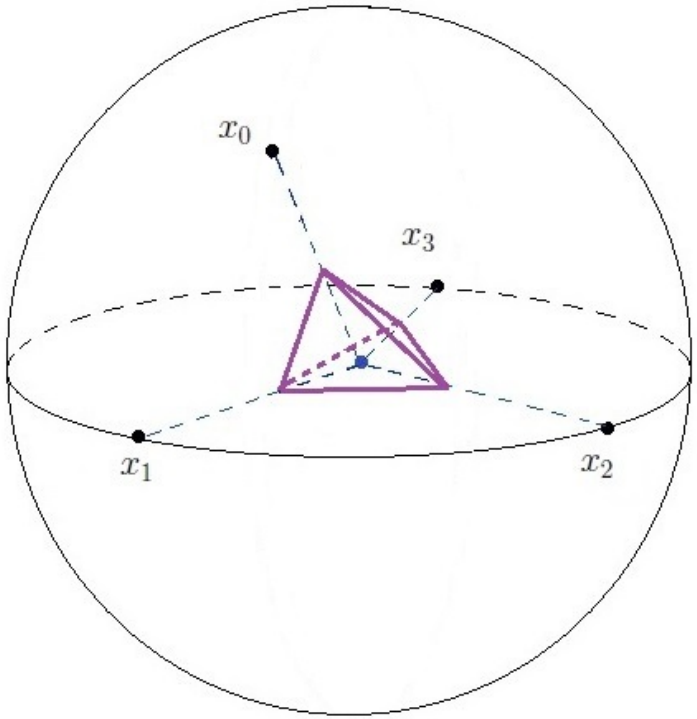
The n-sphere of constant sectional curvature K

The Jacobian of the Blaschke-Petkantschin type change of variables (Miles 1971):

$$J = n!\Delta(x_0, x_1, \ldots, x_n)\left(\frac{\sin(\sqrt{K}R)}{\sqrt{K}}\right)^{n^2-1}$$

The volume of a ball of radius R in $\mathcal{S}^n(K)$:

$$V(R) = \frac{2\pi^{\frac{1}{2}}}{\Gamma(\frac{n}{2})}\int_0^R \sin^{n-1}(t)dt$$



$\Delta(x_0, x_1, \ldots, x_n)$

The n-sphere of constant sectional curvature K

Mean number of vertices

$$\mathbb{E}[N(\mathcal{C})] = E_n - \frac{\mathrm{Sc}}{\lambda^{\frac{2}{n}}} C_n + o(\frac{1}{\lambda^{\frac{2}{n}}})$$

where

- E_n is the mean number of vertices in $\mathbb{R}^n$
- C_n is a positive constant
- $\mathrm{Sc} = n(n-1)K$ is the scalar curvature of $\mathcal{S}^n(K)$

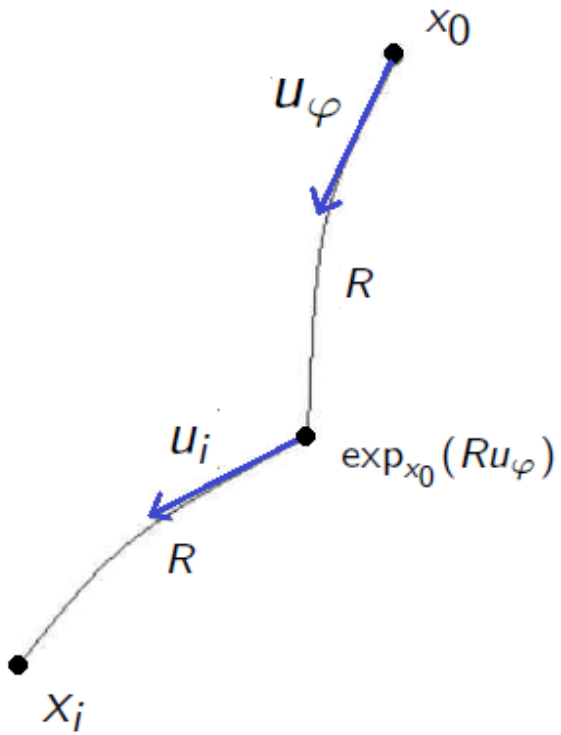
$$E_n = 2\pi^{\frac{n-1}{2}} n^{n-2} \left(\frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n+1}{2})} \right)^n \frac{\Gamma(\frac{n^2+1}{2})}{\Gamma(\frac{n^2}{2})}$$

$$C_n = \frac{2^{1-\frac{2}{n}}}{6n!} \pi^{\frac{n}{2}-\frac{3}{2}} \frac{n^3-2}{(n-1)(n+2)} n^{n+\frac{2}{n}-2} \frac{\Gamma(n+\frac{2}{n})\Gamma(\frac{n}{2})^{n+\frac{2}{n}}\Gamma(\frac{n^2+1}{2})}{\Gamma(\frac{n^2}{2})\Gamma(\frac{n+1}{2})^n}$$

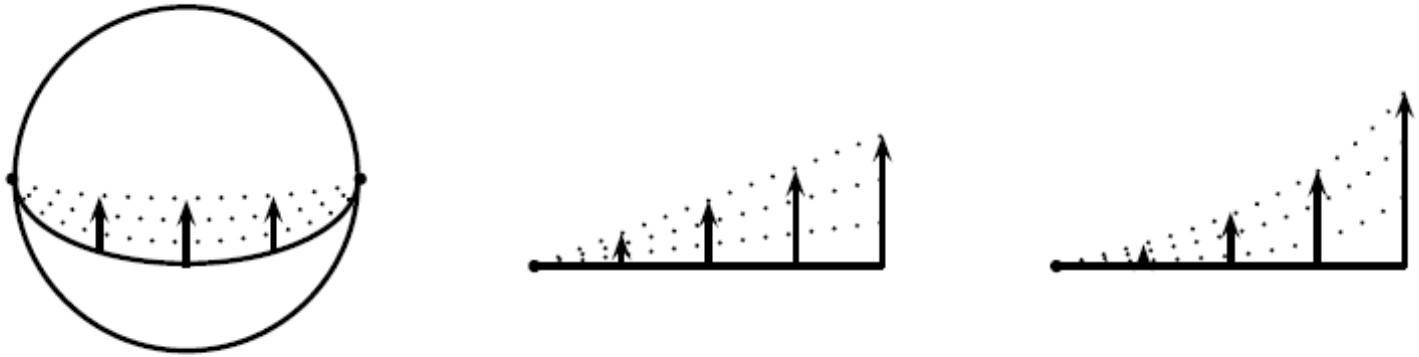
Generalization to a n-manifold M

The Blaschke Petkantschin type change of variables is written as

$$x_i = \exp_{|\exp_{x_0}(Ru_\varphi)}(Ru_i)$$



- The Jacobian of this change of variables involves Jacobi fields



Jacobi fields

- The Jacobian is

$$J = n!\Delta(x_0, x_1, \ldots, x_n) \prod_{i=1}^{n} \prod_{k=1}^{n-1} ||J_i^k(R)|| \cdot \prod_{j=1}^{n-1} ||J_j(R)||$$

- Rauch theorem gives an asymptotic expansion of J

Volume of small geodesic balls

An expansion of the volume of a small geodesic ball on M is given by

$$\mathrm{vol}(\mathcal{B}(z,R)) = \kappa_n R^n \left(1 - \frac{\mathrm{Sc}(z)}{6(n+2)} R^2 + o(R^2)\right)$$

Generalization to a n-manifold M

Mean number of vertices

$$\mathbb{E}[N(\mathcal{C})] = E_n - \frac{\mathrm{Sc}(x_0)}{\lambda^{\frac{2}{n}}} C_n + o(\frac{1}{\lambda^{\frac{2}{n}}})$$

with

- E_n and C_n the same constants as for the sphere
- $\mathrm{Sc}(x_0)$ is the scalar curvature of M at x_0

$$E_n = 2\pi^{\frac{n-1}{2}} n^{n-2} \left(\frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n+1}{2})}\right)^n \frac{\Gamma(\frac{n^2+1}{2})}{\Gamma(\frac{n^2}{2})}$$

$$C_n = \frac{2^{1-\frac{2}{n}}}{6n!} \pi^{\frac{n}{2}-\frac{3}{2}} \frac{n^3-2}{(n-1)(n+2)} n^{n+\frac{2}{n}-2} \frac{\Gamma(n+\frac{2}{n})\Gamma(\frac{n}{2})^{n+\frac{2}{n}}\Gamma(\frac{n^2+1}{2})}{\Gamma(\frac{n^2}{2})\Gamma(\frac{n+1}{2})^n}$$

Euler characteristic and Gauss-Bonnet theorem

S compact surface without boundary

Gauss-Bonnet theorem

$$2\pi\chi(S) = \int_{x\in S} K(x)dx$$

For any graph on S,

$$\chi(S) = V - E + F$$

V: vertices, E: edges, F: faces

Euler characteristic and Gauss-Bonnet theorem

S compact surface without boundary

Gauss-Bonnet theorem

$$2\pi\chi(S) = \int_{x\in S} K(x)dx$$

For any **random** graph on S,

$$\chi(S) = \mathbb{E}[V] - \mathbb{E}[E] + \mathbb{E}[F]$$

V: vertices, E: edges, F: faces

For the Voronoi diagram

In any Voronoi diagram,

- Each vertex is in three cells
- Each edge is in two cells

So we have the relation $3V = 2E$

$$\chi(S) = \mathbb{E}[F] - \frac{1}{2}\mathbb{E}[V]$$

Expression of $\mathbb{E}[V]$ and $\mathbb{E}[F]$

$$3\mathbb{E}[V] = \mathbb{E}[\sum_{C \ \ cell} N(C)] = \lambda \int_{x \in S} \mathbb{E}[N(C(x, \Phi \cup \{x\}))]dx$$

$$\mathbb{E}[N(C(x, \Phi \cup \{x\}))] = 6 - \frac{3K(x)}{\pi\lambda} + o\left(\frac{1}{\lambda}\right)$$

$$\mathbb{E}[V] = 2\lambda \,\mathrm{vol}(S) - \frac{1}{\pi} \int_{x \in S} K(x)dx + o(1)$$

$$\mathbb{E}[F] = \lambda \,\mathrm{vol}(S)$$

Gauss-Bonnet theorem

$$\chi(S) = \frac{1}{2\pi} \int_{x \in S} K(x) dx$$

Take Home Message

- **Dimension 2:**
  - $\hookrightarrow$ Link between mean number of vertices and Gaussian curvature
  - $\hookrightarrow$ Result available for surface of negative curvature (Isokawa 2000)
  - $\hookrightarrow$ Other mean characteristics: area, perimeter
  - $\hookrightarrow$ Elementary proof of Gauss-Bonnet theorem
- **Dimension n:**
  - $\hookrightarrow$ Link between mean number of vertices and scalar curvature
  - $\hookrightarrow$ **Perspectives**: other characteristics to get other curvatures, Gauss-Bonnet type results
  - $\hookrightarrow$ ...

Thank you for your attention!