Metric methods for heteroclinic connections in ODEs and PDEs

Abstract : Let's fix a space X (think at $\mathbb{R}^n$, at least at the beginning) and a double-well potential $W : X \to [0, +\infty]$, with $W(a^+) = W(a^-) = 0$ and $W > 0$ elsewhere. Proving the existence of a curve $u : \mathbb{R} \to X$ minimizing $\int_{\mathbb{R}}(1/2|u'|^2 + W(u))$ with $u(\pm\infty) = a^\pm$ is a classical problem, involving a lack of compactness, due to the interval $\mathbb{R}$ being non-compact and the space $H^1(\mathbb{R})$ being invariant under translations. Many strategies have been proposed to overpass this difficulty and find these curves (called heteroclinic connections, solutions of $u'' = W'(u)$), including some recent ones by N. Alikakos and G. Fusco or by Ch. Sourdis.

A classical approach consists in using the Young inequality and looking rather at $\min k(u)|u'|$, where $k = (2W)^{1/2}$... This new functional is invariant under reparametrization and allows to replace $\mathbb{R}$ with the compact interval $[0, 1]$. In practice, we look for a geodesic in the space X endowed with a weighted distance d_k. The difficulty is that k can vanish, and actually it does.

Starting from ideas that Antonin Monteil (now at UC Louvain) developed during his PhD thesis, I will present joint works with him, where the main point just consists in proving that the new metric space (X, d_k) is proper (closed balls are compact), which implies the existence of geodesics.

A very similar argument to prove the existence of heteroclinic connections has also been used by P. Sternberg and A. Zuniga, by the way. Then, we will pass to the more challenging case where X is itself a functional space, say $L^2(\Omega)$, and $W(u) = \int_\Omega |Du|^2 + w(u)$. This allows to treat problems connected to the PDE $\Delta u = w'(u)$ in cylinders. The most interesting case is $\Omega = \mathbb{R}$, where the problem of connecting two heteroclinic connections via a solution of $\Delta u = w'(u)$ has been treated by S. Alama, L. Bronsard and C. Gui, and by M. Shatzman in different contexts. Of course, we must face extra loss of compactness, but we can solve it, in a different way then the usual strategy (solving the problem on a square and sending the side of the square to ∞), which is what I will try to explain in the talk.

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